ST. ANNE'S COLLEGE OF ENGINEERING & TECHNOLOGY

Subject Name	**:**	**FINITE ELEMENT ANALYSIS**
Subject code	**:**	**ME8692**
Year	**:**	**IIIrd year**
Semester	**:**	**VIth semester**

ME6603 FINITE ELEMENT ANALYSIS L T P C
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OBJECTIVES:

- To introduce the concepts of Mathematical Modeling of Engineering Problems.
- To appreciate the use of FEM to a range of Engineering Problems.

UNIT I INTRODUCTION

Historical Background – Mathematical Modeling of field problems in Engineering – Governing Equations – Discrete and continuous models – Boundary, Initial and Eigen Value problems– Weighted Residual Methods – Variational Formulation of Boundary Value Problems – Ritz Technique – Basic concepts of the Finite Element Method.

UNIT II ONE-DIMENSIONAL PROBLEMS

One Dimensional Second Order Equations – Discretization – Element types- Linear and Higher order Elements – Derivation of Shape functions and Stiffness matrices and force vectors- Assembly of Matrices - Solution of problems from solid mechanics and heat transfer. Longitudinal vibration frequencies and mode shapes. Fourth Order Beam Equation –Transverse deflections and Natural frequencies of beams.

UNIT III TWO DIMENSIONAL SCALAR VARIABLE PROBLEMS

Second Order 2D Equations involving Scalar Variable Functions – Variational formulation –Finite Element formulation – Triangular elements – Shape functions and element matrices and vectors.Application to Field Problems - Thermal problems – Torsion of Non circular shafts – Quadrilateral elements – Higher Order Elements.

UNIT IV TWO DIMENSIONAL VECTOR VARIABLE PROBLEMS

Equations of elasticity – Plane stress, plane strain and axisymmetric problems – Body forces and temperature effects – Stress calculations - Plate and shell elements.

UNIT V ISOPARAMETRIC FORMULATION

Natural co-ordinate systems – Isoparametric elements – Shape functions for iso parametric elements – One and two dimensions – Serendipity elements – Numerical integration and application to plane stress problems - Matrix solution techniques – Solutions Techniques to Dynamic problems – Introduction to Analysis Software.

TOTAL : 45 PERIODS

OUTCOMES:

- Upon completion of this course, the students can able to understand different mathematicalTechniques used in FEM analysis and use of them in Structural and thermal problem

TEXT BOOK:

1. J.N.Reddy, "An Introduction to the Finite Element Method",3rdEdition,Tata McGraw-Hill International,2005.
2. P.Seshu, "Text Book of Finite Element Analysis", Prentice-Hall of India Pvt. Ltd., New Delhi, 2007.

REFERENCES:

1. Rao,S.S.,"The Finite Element Method in Engineering".3rd Edition, Butterworth Heinemann, 2004.
2. Logan,D.L.,"A first course in Finite Element Method",Thomas Asia Pvt.Ltd.,2002.
3. Cook Robert.D., David S Malkus, PleshaMichael.E&Witt,Robert.J., "Concepts and Applications ofFinite Element Analysis",Wiley Student Edition, 2002.
4. Chandrupatla&Belagundu, "Introduction to Finite Elements in Engineering", 3rd Edition, Prentice-Hall college Div,1990.
5. BhattiAsghar M, "Fundamental Finite Element Analysis and Applications", John Wiley & Sons, 2005(Indian Reprint 2013).

UNIT-I

INTRODUCTION

1. **Write short notes on Finite Element Analysis with its historical background.**

Finite Element Analysis:

The Finite Element Analysis is a computer aided mathematical technique that is used to obtain an approximate numerical solution to the fundamental differential and/or integral equations that predict the response of physical systems to external effects.

In Finite Element Analysis, a given domain is viewed as a collection of subdomains, and over each subdomain the governing equation is approximated by any of the traditional vibrational methods.

External influence:

- When a bar is subjected to an axial pull 'P' it elongates
- When a metallic rod is heated its temperature rises
- When a beam is subjected to an external harmonic excitation it vibrates

In the above examples the force 'P', or heat flux 'q' or harmonic excitation force constitute the "external influence" that causes the system to change.

The elongation, temperature rise or vibration represents the system's response to the external influence.

Finite Element Analysis, is a mathematical technique used to predict the response of structures and materials to environmental factors.

Finite Element Analysis (FEA) is used to numerically simulate the real world without the need to test prototypes in a lab.

HISTORICAL BACKGROUND:

Finite Element Analysis was first developed in the early 1960's as a simulation and design tool in the aerospace and nuclear industries where the safety of structures is critical.

The process starts with the creation of a geometric model. Then, the model is subdivided (meshed) into small pieces (elements) of simple shapes connected at specific node points. Within each element, the variation of displacement is assumed to be determined by simple polynomial shape functions and nodal displacements.

Equations for the strains and stresses are developed in terms of the unknown nodal displacements. From this, the equations of equilibrium are assembled in a matrix which can be easily programmed and solved on a computer. After applying the appropriate boundary conditions, the nodal displacements are found by solving the matrix stiffness equation. Once the nodal displacements are known, element stresses and strains can be calculated.

Only within the last few years have computers become powerful enough to solve these FEA math problems in a timely fashion, and thus help improve the engineering process.

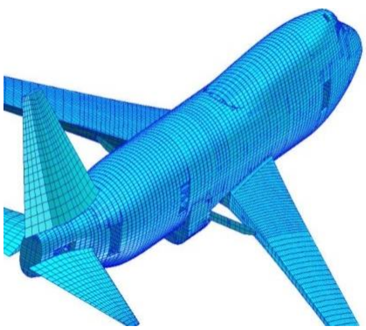
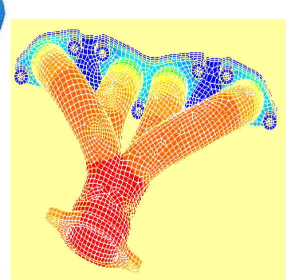
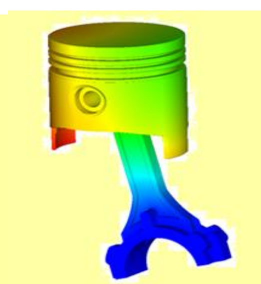
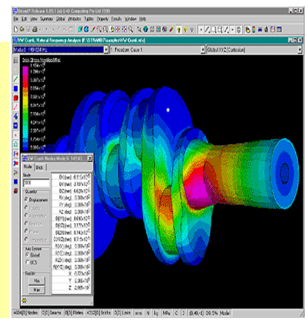
Predictive Method of Analysis Vs Experimental Analysis

- Determination of the solution for a complicated problem by replacing it by a simpler one.

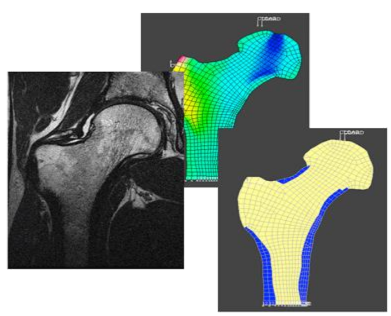
- Geometrically complex domain represented as a collection of smaller manageable domains.
- Solution to these geometrically simple domains is easier.
- Replacing the original complex geometry as an assemblage of smaller simple geometry will result in only an approximate solution.

Applications

➢ Structural Engineering	Aerospace Engineering (a)	Automobile Engineering (b,c,d)
➢ Thermal applications	Acoustics	Flow Problems
➢ Dynamics	Metal Forming	Medical & Dental applications(e)
➢ Soil mechanics etc.		

a) Aeroplane Analysis b) Inlet Manifold c)Piston Analysisd) crankshaft Analysis



e)Elbow analysis

Applications of FEA

Finite element Analysis is Suitable for:

- Complex geometry
- Complex loading
- Complex material properties

2. **Discuss the mathematical modeling of problems in engineering**

MATHEMATICAL MODELING OF PROBLEMS IN ENGINEERING

Any Engineering problems can be solved by using three different methods separately.

1. Analytical method
2. Numerical method
3. Experimental method

Analytical method

- Analytical method is a classical approach technique.
- It gives 100% accurate results
- It is a closed form of solution.
- This method is applicable only for simple problems like cantilever and simply supported beams etc.

Numerical method

- Finite element method is one of the numerical methods.
- Numerical method is a mathematical representation.
- In this method, many assumptions are made and it gives approximate solution.
- This method is applicable even if physical prototype is not available.
- Numerical methods deals with real life complicated problems.
- The result obtained by this method cannot be believed blindly and the results must be verified by the experimental method. This process of comparison is called as **Validation**.
- But this method will be useful to obtain the range of results.
- It minimizes the number of experiments and reduces the cost and time.

NUMERICAL SOLUTION TECHNIQUES

Weighted Residual Methods - Collocation method

- Sub domain method
- Least squares method
- Galerkin method

Finite Difference Method

Rayleigh Ritz Technique

Finite Element Method

Boundary Element Method

Examples:

- Finite element method is useful for linear, non-linear, buckling, thermal, dynamic and fatigue analysis.
- Boundary element method is useful for Acoustics.
- Finite Volume Method is useful for Computational Fluid Dynamics (CFD) and computational electromagnetics.
- Finite Difference Method is used for thermal and fluid flow analysis.

Experimental method

- This method involves in the actual measurements.
- Experimental methods are time consuming and needs expensive setup.
- This method is only applicable only if physical prototype is available.
- Results can be believed but it requires atleast 3 to 5 prototypes readings for testing.
- It uses strain gauge, photo elasticity, vibration measuring instruments, sensors for temperature and pressure measurements, fatigue test, etc.,

Finite Element Method (FEM)

- It is a numerical method.
- Mathematical representation of actual problem.
- It is an approximate method.
- Any continuous object has infinite degrees of freedom and it is just not possible to solve the problem min this format.
- FEM reduces degrees of freedom from infinite to finite with the help of discretization.

Boundary Element Method (BEM)

- It is very powerful and efficient technique to solve acoustics problems.
- Similar FEM, But BEM consider only outer boundary of the surface and volume.
- If the problem is of a volume only outer surfaces are considered if the domain is of area, then only outer periphery is considered.

Based on application, the finite element problems are classified as follows.

I. Structural Problems
II. Non-Structural problems

Structural Problems

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stress and strain in each element can be calculated.

Non-Structural problems

In non-structural problems, temperature or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow etc., for each element can be calculated.

Finite Volume Method (FVM)

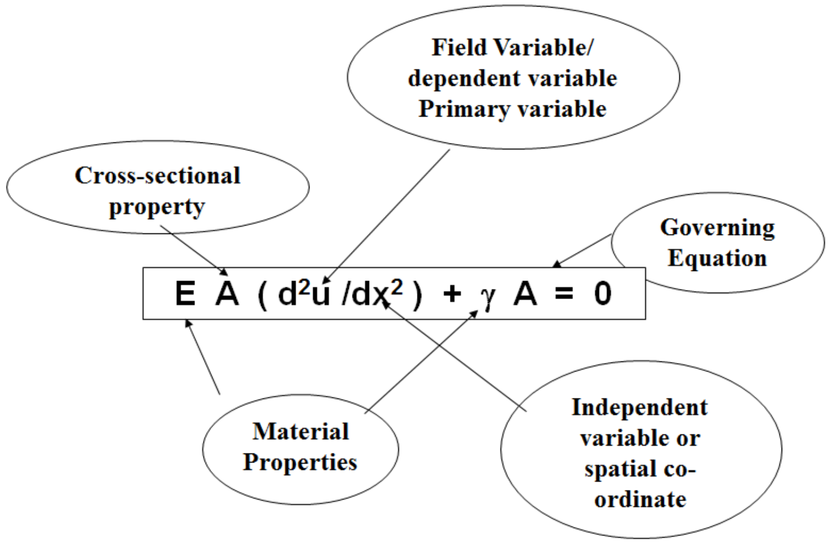
- All CFD (Computational Fluid Dynamics) soft wares are based on Finite Volume Method.
- Unit volume is considered in FVM.
- FVM is based on Navier-Stokes equation (i.e) Mass momentum and enery conservation equilibrium equation.

Finite Difference Method (FDM)

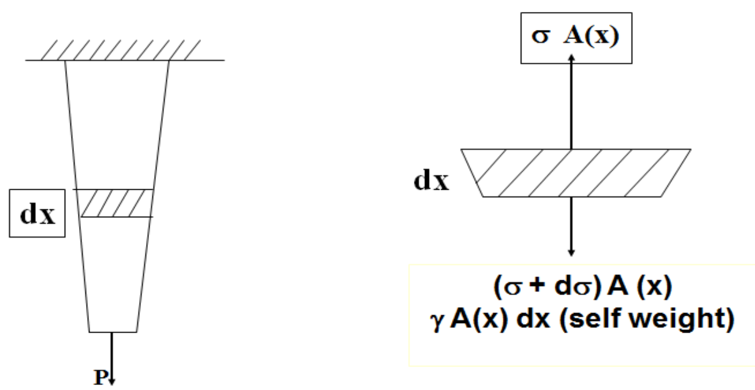
- FDM is used to solve differential equations.
- It uses Taylors series (or) Difference table for converting differential equation into algebraic equation by eliminating higher order terms.
- The various derivatives such as $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ that exist in the differential equation are expressed in an approximate manner as difference form.
- The beam or column is discretized into a number of smaller portions and the end portions are called as nodes or points.
- The differential equation, split in this discrete form is applied to each node.
- This results in a set of simultaneous equations.
- After eliminating some of the unknowns by applying the boundary conditions, the remaining unknowns are solved simultaneously.

4. **Derive the Governing equation of bar element.**

<u>GOVERNING EQUATIONS:</u>Mathematical modeling



Example of a taper rod subjected a point load ‘P’ and its own self weight



For equilibrium $(\sigma + d\sigma)\, A(x) + \gamma\, A(x)\, dx - \sigma\, A(X) = 0$ --(1)

i.e) $d\sigma\, A(x) + \gamma\, A(x)\, dx = 0$ ---(2) $(A(x) = A0 - (A0-A1)\, x/l)$

$$\boxed{\sigma = E\varepsilon = E\frac{du}{dx} \rightarrow (3)}$$

Where σ - stress, ∈ - strain & E - Young’s Modulus (from continuum mechanics, $\in = du / dx$)
(3) in (2) & dividing by dx.

$$\boxed{E\frac{d(\sigma A(x))}{dx} + \gamma A(x) = 0}$$

$$\boxed{E\left(\frac{d\left[A(x)\frac{du}{dx}\right]}{dx}\right)+\gamma A(x)=0\rightarrow(4)}$$

For a bar of constant cross section Governing Equation

$$\boxed{EA(x)\frac{d^2u}{dx^2}+\gamma A(x)=0\rightarrow(5)}$$

Boundary conditions:

1. $U(0) = 0$
2. $\left[EA(x)\ \frac{dU}{dx}\right]_{x=L} = P$

Variables:

- **Primary** eg. Displacement, u Temperature, T
- **Secondary**eg. Force EA du/dx Heat flux –KA dT/dx

Moment – EI (d^2w/dx^2)

Boundary conditions:

- Essential/ Geometric/ Dirichlet Boundary conditions
- Natural/ Force/ Neumann Boundary conditions

Boundary Conditions Can Be Of The Following Two Types

- HOMOGENEOUS eg. u(0)= 0
- NON-HOMOGENEOUS eg. T(0)=80

Loads:

- Volume loads N/m^3 N/meg. Self weight, UDL
- Point loads N

5. **Write short notes on Discrete and continuous models**.

<u>DISCRETE AND CONTINUOUS MODELS</u>

A **discrete element models (DEM)**, also called a **distinct element method** is any of a family of numerical methods for computing the motion and effect of a large number of small particles. Though DEM is very closely related to molecular dynamics, the method is generally distinguished by its inclusion of rotational degrees-of-freedom as well as stateful contact and often complicated geometries (including polyhedra).

With advances in computing power and numerical algorithms for nearest neighbor sorting, it has become possible to numerically simulate millions of particles on a single processor.

Discrete element methods are relatively computationally intensive, which limits either the length of a simulation or the number of particles. Several DEM codes, as do molecular dynamics codes, take advantage of parallel processing capabilities (shared or distributed systems) to scale up the number of particles or length of the simulation. An alternative to treating all particles separately is to average the physics across many particles and thereby treat the material as a continuum.

In the case of solid-like granular behavior as in soil mechanics, the continuum approach usually treats the material as elastic or elasto-plastic and models it with the finite element method or a mesh free method. In the case of liquid-like or gas-like granular flow, the continuum approach may treat the material as a fluid and use computational fluid dynamics. Drawbacks to homogenization of the granular scale physics, however, are well-documented and should be considered carefully before attempting to use a continuum approach.

Continuous modelling is the mathematical practice of applying a model to continuous data (data which has a potentially infinite number, and divisibility, of attributes). They often use differential equations and are converse to discrete modelling.

Modelling is generally broken down into several steps:

- Making assumptions about the data: The modeller decides what is influencing the data and what can be safely ignored.
- Making equations to fit the assumptions.
- Solving the equations.
- Verifying the results: Various statistical tests are applied to the data and the model and compared.
- If the model passes the verification progress it is put into practice.

6. **Derive the difference form for beam element.**

Derivatives in Difference Form:

Frequently deflection and stability problems may involves the integration of the differential equations as given below,

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^2y}{dx^2} + k^2y = 0$$

Solving the above equation by ordinary methods may be either difficult or impossible. The equation may then be made to finite differences where the analytical differential equation is replaced by appropriate finite difference equations.

Usually the finite difference approximations for the first, second, third derivatives, etc., are obtained in two ways, namely

1. Taylor's series expansion
2. Using a difference table

Consider a function

Y=f(x)

Let the x coordinates be divided into equal interval $\Delta x = h = x_i - x_{i-1}$

The corresponding y coordinates y_{i+1}, y_i, y_{i-1}

The difference table is

X	Y	Δy	$\Delta^2 y$
x_{i-1}	y_{i-1}		
x_i	y_i	$y_i - y_{i-1}$	
x_{i+1}	y_{i+1}	$y_{i+1} - y_i$	$y_{i+1} - 2y_i + y_{i+1}$

Frist derivative at point i

For getting the various derivatives at point i,

$\Delta y = y_i - y_{i-1}$ and $\Delta y = y_{i+1} - y_i$

The second derivatives

$\Delta^2 y = (y_{i+1} - y_i) - (y_i - y_{i-1})$

$= (y_{i+1} - 2y_i + y_{i+1})$

w.k.t$\Delta x = h$ this region is subdivided into equal number of small divisions as h.

Now the **Frist derivative can be approximated as**

$\frac{dy}{dx}$at point i = $Limit\ \Delta x \to 0\ \frac{\Delta y}{\Delta x}$

$= \frac{yi+1 - yi}{h}$

$\left(\frac{d^2y}{dx^2}\right) i = \frac{yi+1 - yi}{h}$

Second derivative at point i:

$\left(\frac{d^2y}{dx^2}\right)$at point i = $Limit\ \Delta x \to 0\ \frac{\Delta^2 y}{\Delta x^2}$

$\Delta^2 y = (y_{i+1} - y_i) - (y_i - y_{i-1})$

$= (y_{i+1} - 2y_i + y_{i+1})$

$\left(\frac{d^2y}{dx^2}\right) i = \frac{y_{i+1} - 2y_i + y_{i+1}}{h^2}$

7. **Problem:**

Determine the deflection under load using finite difference method by sub dividing the length of beam into two region and four regions. The beam is simply supported at the ends with a center point load with uniform section. Exploit symmetry involved in the problems. What is your interference concerning accuracy as discretization becomes finer and finer.

Given:

Divide the beam into two elements

Boundary condition at nodal points

y1=0

y3=0

We know

$\left(\frac{d^2y}{dx^2}\right) = -\frac{M}{EI}$ ……(1)

$EI\left(\frac{d^2y}{dx^2}\right) = -M$

To obtain y_2, use Difference equation at point (2)

$\left(\frac{d^2y}{dx^2}\right) i = \frac{y_1 - 2y_2 + y_3}{h^2}$ & M = $\frac{Wl}{4}$ ……(2)

Substitute values of $\left(\frac{d^2y}{dx^2}\right)$& M in Eqn. (1)

$$\frac{y_1 - 2y_2 + y_3}{h^2} = -\frac{Wl}{4} X \frac{1}{EI}$$

Substitute boundary condition at Eqn. (2) $\left(h^2 = \frac{l^2}{4}\right)$

(2) $\frac{0-2y_2+0}{\frac{l^2}{4}} = -\frac{Wl}{4} X \frac{1}{EI}$

$$2y_2 = \frac{Wl}{4} X \frac{1}{EI} X \frac{l^2}{4}$$

Deflection at node 2 $= y_2 = \frac{W}{32}\frac{l^3}{EI}$

Case (ii) Divide the beam into more elements.(4 elements)

At Node: 2

$$\frac{y_1 - 2y_2 + y_3}{h^2} = -\frac{Wl}{8EI}$$

Apply Boundary condition ($y_1 = 0$ &$h=\frac{l}{4}, M = -\frac{W}{8}\frac{l}{EI}$)

$$\frac{-2y_2 + y_3}{\left(\frac{l}{4}\right)^2} = -\frac{W}{8}\frac{l}{EI}$$

$-2y_2 + y_3 = -\frac{W}{8}\frac{l}{EI} x \frac{l^2}{16}$ …….. (3)

At Node 3:

$$\frac{y_2 - 2y_3 + y_4}{h^2} = -\frac{Wl}{4EI}$$

Apply $h = \frac{l}{4}$($y_2 = y_4$)

$$\frac{-2y_2 + 2y_3}{\left(\frac{l}{4}\right)^2} = \frac{-W}{4}\frac{l}{EI}$$

$-2y_2 + 2y_3 = \frac{-W}{4}\frac{l}{EI} x \frac{l^2}{16}$

Solving equation (3) and (4), $-y_3 = \frac{-W}{64}\frac{l^3}{EI} - \frac{W}{128}\frac{l^3}{EI}$

$y_3 = \frac{3W}{128}\frac{l^3}{EI}$

Substituting y_3 in equation (4)

$y_3 = \frac{W}{42.7}\frac{l^3}{EI}$

$2y_2 - 2x\frac{3W}{128}\frac{l^3}{EI} - \frac{W}{64}\frac{l^3}{EI}$

$2y_2 = \frac{-W}{64}\frac{l^3}{EI} + \frac{6W}{128}\frac{l^3}{EI}$

$2y_2 = \frac{6Wl^3 - 2Wl^3}{128EI}$

$y_2 = \frac{Wl^3}{64EI}$

Therefore value of y_3 at Node 3 is nearer to exact value.

i.e $y_3 = \frac{W}{42.7}\frac{l^3}{EI}$

the exact value at node 3 is

$$\frac{W}{48}\frac{l^3}{EI}$$

This shows that as the number of element increases, the answer approaches to exact value.

8. Problem :

A simple beam of span 5m carries a point load of 10kN mid span. Find the deflection at mid span using Macaulay's method. (Nov/Dec 2013)

Solution:

(i) Find reactions at A and B

Taking moment about A; R_B x 5-10 x 2.5=0

R_B =5kN; Also R_A+ R_B**=10**

R_B =10-5=5kN;

Consider a section X-X at distance x from B.

Bending moment at X-X

$- M_x = EI\left(\frac{d^2y}{dx^2}\right) = R_B\,x - W(x\text{-}L/2) = 5x2\text{-}10x(x\text{-}\frac{5}{2})$

$EI\left(\frac{d^2y}{dx^2}\right) = 5x\text{-}10(x\text{-} 2.5)$ ……(1)

On first integration we get slope equation.

$EI\left(\frac{dy}{dx}\right) = \frac{5x^2}{2} + \frac{-10(x-2.5)^2}{2} + C_1$

On second Integration the get deflection equation

$EI\ y = \frac{5x^3}{6} + C_1x - \frac{10(x-2.5)^3}{6} + C_2$

Boundary Condition:

(i) When x=2.5 m; $\frac{dy}{dx} = 0$

(ii) When x=0;y=0

(iii) Applying boundary condition in (2) and (3)

(2) $0 = \frac{5x2.5^2}{2} - 0 + C_1$

$C_1 = -15.625$

(3) $0 = \frac{(0)^3}{6} + C_1x0 + C_2$

$C_2 = 0$

The deflection at x=2.5m

$Ely = \frac{5(x)^3}{6} + C_1x$

$y = \frac{1}{EI}(\frac{5(2.5)^3}{6} + (-15.625x2.5)$

$y = \frac{1}{E2}(\frac{5(2.5)^3}{6} + (-15.625x2.5)$

$y = \frac{1}{2X10^4}(-26.042)$

$= -1.302X10^{-3}m$

=-1.302mm

Here, this strength of material problem is solved by using analytical method easily.

Linear equation: displacement, $u = a_0 + a_1x$

(Or)

Quadratic equation:

Displacement, $u=a_0+ a_1x+a_2x^2$

By determining the unknown parameters a_0, a_1 and a_2 we can find a solution with minimum error by applying boundary conditions.

9. **Write short notes on Boundary, initial and eigen value problem.**

BOUNDARY, INITIAL AND EIGEN VALUE PROBLEMS:

Boundary Value Problem (BVP)

A boundary value problem is one where the field variable (e.g., temperature or displacement) and possibly its derivatives are required to take on specified values on the boundary

e.g., KA dT / dx = Q,

where K= Thermal conductivity,

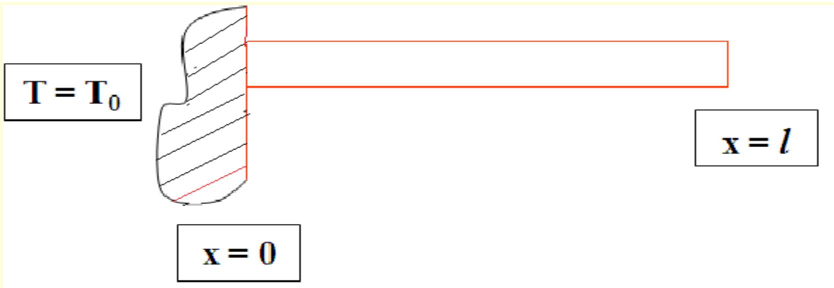
A = area of cross-section,

Q = Heat flux.

$$-\frac{d}{dx}\left[KA(x)\frac{dT(x)}{dx}\right] + hp\,[T(x) - T_\infty] = 0$$

Boundary conditions: @ x = 0, $T = T_0$

@ x = l, -KA (dt/dx) = 0



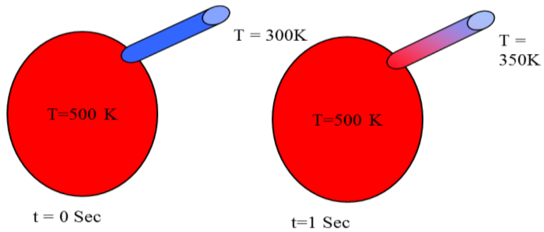
Initial Value Problem (IVP)

An Initial value problem is one where the field variable and possibly its derivatives are specified initially (i.e., at time t=0). These are generally time dependent problems.

Examples include: Unsteady heat conduction, Dynamic problems

Initial conditions: @ time $t = 0$

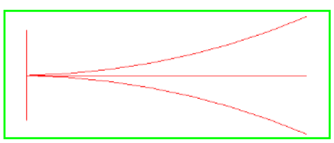
i) $du/dt = C_0$ where Velocity = du /dt ii) displacement $u = a_0$



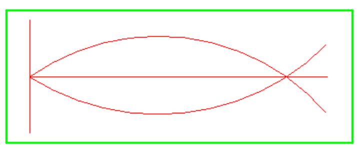
Eigen Value Problem (EVP)

An eigen value problem is one where the problem is defined by a homogeneous differential equation that is one where the right hand side is zero. An important class of eigen value problems is the ‘Vibration of Beams” or continuous systems.

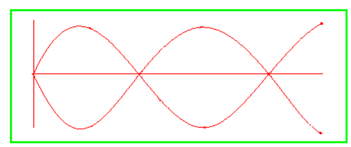
First mode shape



Second mode shape



Third mode shape



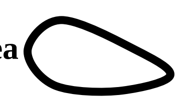
Dimensionality

Physical problems can be classified into

(i) I dimensional

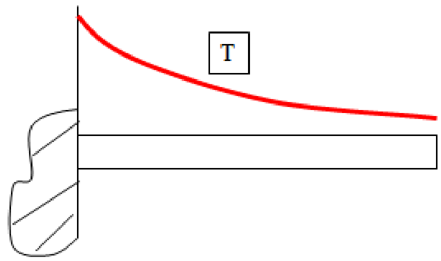
(ii) II dimensional

(iii) III dimensional problems

Domain	**Geometry**	**Boundary**
1D	**Line**	**Points**
2D	**Area** 	**Curves**
3D	**Volume** 	**Area**

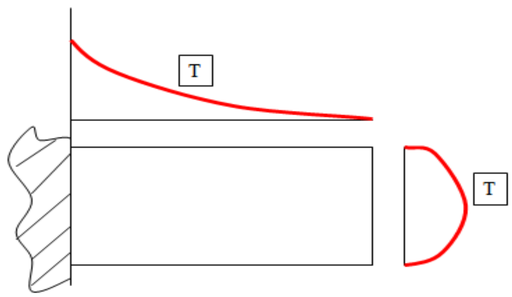
I-D PROBLEMS:-

When the geometry, material properties and field variables such as displacement, temperature, pressure etc can be described in terms of only one spatial co-ordinate we can go in for one-dimensional modeling



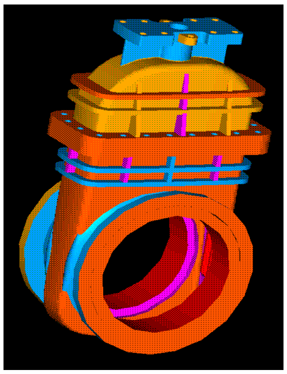
2D PROBLEMS:-

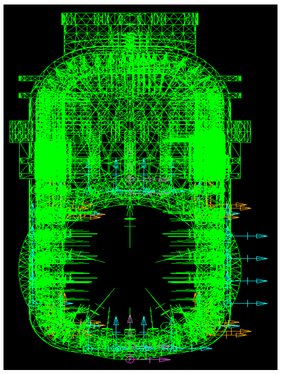
When the geometry and other parameters are described in terms of two independent co-ordinates we go in for two-dimensional modeling.



3D PROBLEMS:-

If the geometry, material properties and other parameters of the body can be described by three independent spatial co-ordinates, we can discretize the body using 3 dimensional modeling.





Exact and approximate solutions:

- An exact solution satisfies the differential equation at every point in the domain and the boundary conditions on the boundary
- An approximate solution satisfies the boundary conditions completely and as closely as possible the differential equation

$$\boxed{E\left(\frac{d\left[A(x)\frac{du}{dx}\right]}{dx}\right)+\gamma A(x)=0}$$

$$E\left(\frac{d\left[A(x)\frac{d\bar{u}}{dx}\right]}{dx}\right)+\gamma A(x)=R$$

NUMERICAL SOLUTION OF BVPs

(i) Choose a trial solution U(x) for U(x)

(ii) Select a criterion for minimising the error

U(x) can be a trigonometric function such as Asinx

or a logarithmic function log x

or a hyperbolic function

or polynomial functions

$$\overline{U}(x)=a_1+a_2\,x+a_3x^2+a_4\,x^3$$

1. **Methods of weighted residuals** (WRM) which are applicable when the governing equations are differential equations.
2. **Ritz variational method** which is applicable when the governing equations are variational (integral) equations with an associated quadratic functional.

10. Explain in detail about the procedure of Weighted residual method.

<u>WEIGHTED RESIDUAL METHOD:</u>

It is a powerful approximate procedure applicable to several problems. For non – structural problems, the method of weighted residuals becomes very useful.

The WRM is a generalization of Ritz method.

The WRM criteria seek to minimise the error involved in not satisfying the governing differential equations.

The most popular methods are

(i) The Collocation method (or) Point Collocation Method

(ii) The Sub -Domain method (or) Sub -Domain Collocation method

(iii) The Least squares method.

(iv) The Galerkin method.

General Procedure for solving weighted residual method

Step 1: Assume the trial solution as y

Step 2: Reconstruct the trial solution in terms of trial function by applying the boundary conditions.

Step 3: Obtain the residual function by back substituting the reconstructed trial solution the differential equation.

Step 4: Select the solution criteria.

Collocation Method

For each undetermined coefficient a_i, choose a point x_i in the domain and at each such point called as collocation point force the residual to be exactly zero

R=0

ie. The collocation points may be located anywhere on the boundary or in the domain.

Residuals are set to zero at n different locations Xi, and the weighting function w.

δ W R (x_i; a_1, a_2, $a_3 \ldots a_n$) dx = 0

Where

δ---- Domain

R---- Residual

W----Weighing function

The weighing function W is denoted as $\delta(x-x_i)$

$W=\delta(x-xi)$

xi ---- collocation points and are selected by discretization of analyst.

So,

$\int \delta\,(x - xi)=1$

And

$R\,(x_i; a_1, a_2, a_3\ldots a_n) = 0$

The chosen points are called collocation points. They may be located any were on the boundary or in the domain.

The Sub-Domain Method

For each undetermined parameter choose an interval Δx, in the domain. Then force average of the residual in each interval to be zero.

$$\frac{1}{\Delta x_1}\int_{\Delta x_1} R(x)\,dx = 0$$

$$\frac{1}{\Delta x_2}\int_{\Delta x_2} R(x)\,dx = 0$$

$$\frac{1}{\Delta x_n}\int_{\Delta x_n} R(x)\,dx = 0$$

Here the weighing function is made unity over a portion of the domain and zero elsewhere.

So,

$$W_1=\begin{cases} 1\ for\ x\ in\ D_1 \\ 0\ for\ x\ not\ in\ D_1 \end{cases}$$

$$W2=\begin{cases} 1\ for\ x\ in\ D_2 \\ 0\ for\ x\ not\ in\ D_2 \end{cases}$$

$$W_n=\begin{cases} 1\ for\ x\ in\ D_n \\ 0\ for\ x\ not\ in\ D_n \end{cases}$$

Where D---- Domain

LEAST SQUARES TECHNIQUE:

In this method we minimize with respect to each undetermined coefficient the integral of the square of the residue over the entire domain

$$\partial / \partial \ a_1 \int_1^2 R^2 \text{ (x) dx} = 0$$

$$\int_1^2 R(x)(\partial \text{R}/\partial \bar{a}_1) \text{ dx} = 0$$

∫ [R (x; a1, a2, a3… an)]2 dx = minimum.

THE GALERKIN METHOD

For each undetermined parameter we require that a weighted average of R(x) over the entire domain be zero. The weighting functions are the trial functions associated with the generalised coefficients

$$\int_1^2 R(x)\, \phi_i \text{ (x) dx} = 0$$

wi = Ni (x)

Ni (x) ----- Trial function

∫ Ni (x) [R (x; a1, a2, a3… an)]2 dx = 0, i = 1, 2, 3, …n.

GENERAL WRM

$$\int_\Omega R(X)\, w_i \text{ (x) dx} = 0$$

- The Collocation method - **dirac delta function**
- The Sub-Domain method - **Unity**
- The Least squares method - **Residue**
- The Galerkin method – **coefficient of the undetermined coefficients in the trial solution**

11. The governing differential equation for the fully developed laminar flow is given by $\mu\frac{d^2y}{dx^2}$+ρg cosɵ=0. If boundary conditions are $\frac{du}{dx}_{x=0}$ =0, u (L) =0. Find the velocity distribution, u (x).

Given: $\mu\frac{d^2y}{dx^2}+\rho g\cos\theta=0$

Boundary condition : $\frac{du}{dx}_{x=0}$ **=0, u (L) =0**

To find: velocity distribution.

Solution:

Assume trial function $u(x) = a_0 + a_1x + a_2x^2$ ----1

First boundary condition – du/dx = at x = 0

$du/dx = a_1 + 2a_2x$

at x= 0; du/dx = 0,

$a_1 = 0$

Second boundary condition;

x=L, u(x) = 0

$u(x) = a_0 + a_1x + a_2x^2$

$a_0 + a_1L + a_2L^2 = 0$

sub $a_1 = 0$

$a_0 + a_2L^2 = 0$

$a_0 = -a_2L^2$

sub a_0 and a_1 value in equation 1

$u(x) = -a_2L^2 + 0 + a_2x^2$

$u(x) = a_2 [x^2 – L^2]$ --- 2

$du/dx=a_2[2x]$

$d^2u/dx^2=a_2$

W.K.T

Residual, R = $\mu\frac{d^2y}{dx^2}+\rho g\cos\theta=0$

$\mu(2a_2) + \rho g\cos\theta = 0$

$2\mu(a_2) = - \rho g\cos\theta$

$a_2 = -\rho g\cos\theta / 2\mu$

sub a_2 value in eqn 2

$u(x) = \rho g\cos \theta / 2\mu\ [L^2 - x^2]$

velocity distribution; $u(x) = \rho g\cos \theta / 2\mu\ [L^2 - x^2]$

12. Find the solution for the following differential equation.

$$\mathbf{E\ I\ \frac{d^4u}{dx^4} - q_0 = 0.}$$

The boundary conditions are u (0)=0, $\frac{du}{dx}$ (0) =0,

$\frac{d^2u}{dx^2}$(L)=0, $\frac{d^3y}{dx^3}$(L)=0L) (April/May 2011)

Given:

The governing differential equation:

$$E\ I\ \frac{d^4u}{dx^4} - q_0 = 0.$$

Boundary condition:u (0)=0, $\frac{du}{dx}$ (0) =0,

$\frac{d^2u}{dx^2}$(L)=0, $\frac{d^3y}{dx^3}$(L)=0L)

Solution:

Assume trial function, let $u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots\ldots$ 1

First boundary condition at x = 0, u(x) = 0

$a_0+0+0+0 = 0$

$a_0= 0$

Second boundary condition, x=0, du/dx = 0.

$du/dx = 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$

$a_1+0+0+0 = 0$

$a_1 = 0$

Third boundary condition, x = L, $d^2u/dx^2 = 0$.

$d^2u/dx^2= 2a_2 + 6a_3x + 12a_4x^2$

$0 = 2a_2 + 6a_3L + 12a_4L^2$

$a_2 = -[3a_3L + 6a_4L^2]$

Fourth boundary condition, x = L, $d^3u / dx^3 = 0$

$d^3u / dx^3 = 0 + 6a_3 + 24\ a_4\ x$

$0 = 6a_3 + 24\ a_4\ x$

$6a_3 = -24a_4L$

$a_3 =-4a_4L$

sub a_0, a_1, a_2 and a_3 value in eqn 1

$u(x) = 0 + 0 – [3a_3L + 6a_4L^2]x^2 – 4a_4Lx^3 + a_4x^4$

$= - [3a_3L + 6a_4L^2]x^2 – 4a_4Lx^3 + a_2x^4$

$= - [3(-4a_4L)L + 6a_4L^2]x^2 – 4a_4Lx^3 + a_4x^4$

$= 12a_4L^2x^2 – 6a_4L^2x^2 – 4a_4Lx^3 + a_4x^4$

$= a_4[12L^2x^2 – 6L^2x^2 – 4Lx^3 + x^4]$

$=a_4[6L^2x^2 – 4Lx^3 + x^4]$

$u(x) = a_4[6L^2x^2 – 4Lx^3 + x^4]$ -----2

$du/dx = a_4[12L^2x – 12Lx^2 + 4x^3]$

$d^2u/dx^2 = a_4[12L^2 – 24Lx + 12x^2]$

$d^3u/dx^3 = a_4[24x-24L]$

$d^4u/dx^4 = 24a_4$

W.K.T

Residual R = $E I \frac{d^4u}{dx^4} -q_0=0.$

$24EIa_4 – q_0 = 0$

$a_4 = q_0 / 24EI$

Sub a_4 value in eqn 2

$U(x) = q_0/24EI[x^4-4Lx^3+6L^2x^2]$

13. The following differential equation is available for a physical phenomenon

$$A E \frac{d^2u}{dx^2}+a x=0$$

The boundary conditions are u (0)=0, A E $\frac{du}{dx}$ $_{x=L}$ =0. (Nov/Dec 2013)

Given:A E $\frac{d^2u}{dx^2}$+a x=0

Boundary condition: u (0)=0, A E $\frac{du}{dx}$ $_{x=L}$ =0

Solution:

Assume trial function, let $u(x) = a_0+ a_1x + a_2x^2 + a_3x^3$ ----- 1

First boundary condition, x= 0, u(x) = 0

$a_0+0+0+0 = 0$

$a_0 = 0$

second boundary condition, x = L, AE du / dx = 0

$du/dx = 0 + a_1 + 2a_2x + 3a_3x^2$

$a_1 = -(2a_2L + 3a_3L^2)$

sub a_0& a_1 value in eqn 1,

$u(x) = 0 + - (2a_2L + 3a_3 L^2)x + a_2x^2 + a_3x^3$

$= -2a_2Lx – 3a_3L^2x + a_2x^2 + a_3x^3$

$= a_2[x^2 – 2Lx] + a_3[x^3 – 3L^2x]$

$u(x) = a_2[x^2 – 2Lx] + a_3 [x^3 – 3L^2x]$----2

W.K.T

Residual R = A E $\frac{d^2u}{dx^2}$+a x=0—3

$du/dx = a_2[2x-2L] + a_3[3x^2 – 3L^2]$

$d^2u/dx^2 = 2a_2 + 6a_3x$

Sub d^2u/dx^2 value in eqn 3

$R = AE(2a_2 + 6a_2x) + ax$ - - 4

From Galerkin's technique

$$\int_0^L w_i R dx = 0 \quad --- 5$$

From eqn 2 w.k.t

$w_1 = (x^2 – 2Lx)$; $w_2 = (x^3 – 3L^2x)$

Sub w_1& w_2 value in eqn 5

$$\int_0^L \{(x^2 - 2Lx)[AE(2a_2 + 6a_3x) + ax] + (x^3 - 3L^2x)[AE(2a_2 + 6a_3x) + ax]\}dx = 0$$

Integrating the above eqn with respect to x

$$\int_0^L \begin{bmatrix} 2AEa_2x^2 + 6AEa_3x^3 + ax^3 - 4AEa_2Lx - 12AEa_3Lx^2 - 2aLx^2 + 2AEa_2x^3 + \\ 6AEa_3x^4 + ax^4 - 6AEa_2L^2x - 18AEa_3L^2x^2 - 3aL^2x^2 \end{bmatrix} = 0$$

$$\left[\begin{array}{l} 2AEa_2\dfrac{x^3}{3}+6AEa_3\dfrac{x^4}{4}+a\dfrac{x^4}{4}-4AEa_2L\dfrac{x^2}{2}-12AEa_3L\dfrac{x^3}{3}-2aL\dfrac{x^3}{3}+2AEa_2\dfrac{x^4}{4}+ \\ 6AEa_3\dfrac{x^5}{5}+a\dfrac{x^5}{5}-6AEa_2L^2\dfrac{x^2}{2}-18AEa_3L^2\dfrac{x^3}{3}-3aL^2\dfrac{x^3}{3} \end{array}\right]_0^L=0$$

$$\left[\begin{array}{l} 2AEa_2\dfrac{L^3}{3}+6AEa_3\dfrac{L^4}{4}+a\dfrac{L^4}{4}-4AEa_2L\dfrac{L^2}{2}-12AEa_3L\dfrac{L^3}{3}-2aL\dfrac{L^3}{3}+2AEa_2\dfrac{L^4}{4}+ \\ 6AEa_3\dfrac{L^5}{5}+a\dfrac{L^5}{5}-6AEa_2L^2\dfrac{L^2}{2}-18AEa_3L^2\dfrac{L^3}{3}-3aL^2\dfrac{L^3}{3} \end{array}\right]-$$
$$\left[\begin{array}{l} 2AEa_2\dfrac{0^3}{3}+6AEa_3\dfrac{0^4}{4}+a\dfrac{0^4}{4}-4AEa_2L\dfrac{0^2}{2}-12AEa_3L\dfrac{0^3}{3}-2aL\dfrac{0^3}{3}+2AEa_2\dfrac{0^4}{4}+ \\ 6AEa_3\dfrac{0^5}{5}+a\dfrac{0^5}{5}-6AEa_2L^2\dfrac{0^2}{2}-18AEa_3L^2\dfrac{0^3}{3}-3aL^2\dfrac{0^3}{3} \end{array}\right]=0$$

$$\left[\begin{array}{l} \dfrac{2}{3}AEa_2L^3+\dfrac{3}{2}AEa_3L^4+\dfrac{aL^4}{4}-2AEa_2L^3-4AEa_3L^4-\dfrac{2}{3}aL^4+\dfrac{1}{2}AEa_2L^4+ \\ \dfrac{6}{5}AEa_3L^5+\dfrac{aL^5}{5}-3AEa_2L^4-6AEa_3L^5-aL^5 \end{array}\right]=0$$

$$AEa_2L^3\left[\frac{2}{3}-2+\frac{L}{2}-3L\right]+AEa_3L^4\left[\frac{3}{2}-4+\frac{6L}{5}-6L\right]+aL^4\left[\frac{1}{4}-\frac{2}{3}+\frac{L}{5}-L\right]=0$$

$$AEa_2L^3\left[\frac{-8-15L}{6}\right]+AEa_3L^4\left[\frac{-25-48L}{10}\right]+aL^4\left[\frac{-25-48L}{60}\right]=0$$

Solving eqn we get

$a_3 = -a / 6AE$

sub a_3 value in eqn we get

$a_2 = 0$

sub a_2& a_3 value in eqn – 2

$u(x) = a/6AE\ [\ 3L^2x – x^3]$

14. The governing differential equation for the long cylinder of radius R with heat generation q_0 is given by

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{q_0}{k}=0$$

The boundary conditions are T (R) = T_w

$$q_0\pi R^2L= (-k)\ (2\pi\ R\ L)\ \frac{dT}{dr}_{r=R}$$

Find the temperature distribution T as a function of radial location r.

Given:

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{q_0}{k}=0$$

Boundary condition: $q_0\pi R^2L = (-k)(2\pi R L)\frac{dT}{dr}_{r=R}$

Temperature distribution T(r),

Solution:

Assume trial function, $T = a_0 + a_1(r - R) + a_2(r - R)^2$ ----1

First boundary condition, r =R, $T = T_w$

$T_w = a_0 + a_1(R-R) + a_2(R-R)^2$

$a_0 = T_w$

Second boundary condition: $q_0\pi R^2L = (-k)(2\pi R L)\frac{dT}{dr}_{r=R}$

$dT/dr = 0 + a_1(1-0) + a_2 2(r-R)(1-0)$

$= a_1 + 2a_2(r-R)$

$= a_1 + 2a_2(R-R)$

$= a_1$

W.K.T

$-k2\pi RL\, dT/dr = q_0\pi R^2L$

$-k2\pi RL(a_1) = q_0\pi R^2L$

$a_1 = -q_0R/2k$

sub a_0 & a_1 in eqn 1

$T = T_w - q_0R / 2k\,(r - R) + a_2(r - R)^2$ -----2

W.K.T, residual $R = \frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{q_0}{k}=0$ ----3

$dT/dr = 0 - (q_0R/2k) + a_2 2(r-R)$

$d^2T/dr^2 = 0 + 2a_2(1 - 0)$

$= 2a_2$

$R = 2a_2 + q_0/k + 1/r\,(-q_0R/2k + 2a_2(r-R))$ -----4

Galerkin's technique

$$\int_0^L w_i R\,dr = 0$$

$$2\pi L\int_0^R (r-R)^2[2a_2r + q_0r/k - q_0R/2k + 2a_2r - 2a_2R]dr = 0 \text{ ----5}$$

$$2\pi L\int_0^R (r-R)^2\left[2a_2+\frac{q_0}{k}+\frac{q_0R}{2kr}+\frac{2a_2}{r}(r-R)\right]rdr=0$$

$$2\pi L\int_0^R (r-R)^2\left[2a_2r+\frac{q_0r}{k}+\frac{q_0R}{2k}+2a_2(r-R)\right]dr=0 \qquad \text{------------5}$$

Apply Bernoull's formula:

$$\int uvdx = uv_1 + u'v_2 + u''v_3 - u'''v_4 + ...$$

u – differentiate; v – integrate

u = $(r – R)^2$

v = $2a_2r+\frac{q_0r}{k}+\frac{q_0R}{2k}+2a_2(r-R)$

Differentiating u with respect to r,

u – $(r\text{-}R)^2$; u' – 2(r-R); u" -2; u'" -0.

Integrating v with respect to r,

v = $2a_2r+\frac{q_0r}{k}+\frac{q_0R}{2k}+2a_2(r-R)$

v1 = $2a_2\frac{r^2}{2}+\frac{q_0r}{k}\frac{r^2}{2}+\frac{q_0Rr}{2k}+2a_2\frac{r^2}{2}-2a_2Rr$

v2 = $a_2\frac{r^3}{3}+\frac{q_0r}{2k}\frac{r^3}{3}+\frac{q_0R}{2k}\frac{r^2}{2}+2a_2\frac{r^3}{3}-2a_2R\frac{r^2}{2}$

v3 = $a_2\frac{r^4}{12}+\frac{q_0r}{2k}\frac{r^4}{12}+\frac{q_0R}{2k}\frac{r^3}{6}+2a_2\frac{r^4}{12}-2a_2R\frac{r^3}{6}$

Now sub the u & v value in eqn we get

$$\int uvdx = uv_1 + u'v_2 + u''v_3 - u'''v_4 + ...$$

$$=(r-R)^2\left[a_2r^2+\frac{q_0r^2}{2k}-\frac{q_0Rr}{2k}+a_2r^2-2a_2Rr\right]-$$
$$2(r-R)\left[\frac{a_2r^3}{3}+\frac{q_0r^3}{6k}-\frac{q_0Rr^2}{4k}+\frac{a_2r^3}{3}-a_2Rr^2\right]$$
$$+2\left[\frac{a_2r^4}{12}+\frac{q_0r^4}{24k}-\frac{q_0Rr^3}{12k}+\frac{a_2r^4}{12}-\frac{a_2Rr^3}{3}\right]$$

Sub the value in eqn 5 we get

$$2\pi L\left[\begin{array}{l}(r-R)^2\left[a_2 r^2+\frac{q_0 r^2}{2k}-\frac{q_0 Rr}{2k}+a_2 r^2-2a_2 Rr\right]- \\ 2(r-R)\left[\frac{a_2 r^3}{3}+\frac{q_0 r^3}{6k}-\frac{q_0 Rr^2}{4k}+\frac{a_2 r^3}{3}-a_2 Rr^2\right] \\ +2\left[\frac{a_2 r^4}{12}+\frac{q_0 r^4}{24k}-\frac{q_0 Rr^3}{12k}+\frac{a_2 r^4}{12}-\frac{a_2 Rr^3}{3}\right]\end{array}\right]_0^R=0$$

$$\frac{-a_2 R^4}{3}=\frac{q_0 R^4}{12k}$$

$$a_2=\frac{-3q_0 R^4}{R^4 * 12k}$$

$a_2 = -q_0/4k$

sub a_2 value in eqn 2,

$$T=T_w-\frac{q_0 R}{2k}(r-R)+\frac{-q_0}{4k}(r-R)^2$$

$$=T_w-\frac{q_0 Rr}{2k}+\frac{q_0 R^2}{2k}-\frac{-q_0 r^2}{4k}-\frac{q_0 R^2}{4k}+\frac{q_0 2rR}{4k}$$

$$=T_w-\frac{q_0 Rr}{2k}+\frac{q_0 R^2}{2k}-\frac{-q_0 r^2}{4k}-\frac{q_0 R^2}{4k}+\frac{q_0 rR}{2k}$$

$$=T_w+\frac{q_0 R^2}{2k}\left[\frac{1}{2}-\frac{1}{4}\right]-\frac{q_0 r^2}{4k}$$

$T-T_w = q_0/4k[R^2-r^2]$

15. The differentials equation of a physical phenomenon is given by, $\frac{d^2y}{dx^2}$+500x^2=0, 0≤x≤1 Trial function, y=a$_1$(x-x^4), boundary conditions are, y (0)=0; y (1)=0 Calculate the value of the parameter a$_1$ by the following methods: (i) Point collocation; (ii) subdomain collocation; (iii) least squares; (iv) galerkin.(Nov/Dec 2009)

Given: $\frac{d^2y}{dx^2}+500x^2=0$ ------1

Boundary condition – y(0) = 0; y(1) =0

Trial function - $y=a_1(x-x^4)$

To find: (i) Point collocation;(ii) subdomain collocation; (iii) least squares; (iv) galerkin.

First boundary condition; x = 0, y = 0; a_1 = 0;

Second boundary condition; x= 1, y = 0, a_1 = 0

<u>Point collocation method:</u>

$y = a_1 (x - x^4)$

$dy/dx = a_1(1 - 4x^3)$

$d^2y/dx^2 = -12a_1x^2$

sub d^2y/dx^2 value in eqn – 1

$R = -12a_1x^2 + 500x^2$ - - - - 2

Let sub x = 1/2 in eqn 2

$R = -12a_1(1/2)^2 + 500(1/2)^2 = 0$

$-12a_1(1/4) + 500(1/4) = 0$

$-3a_1 + 125 = 0$

$a_1 = 41.66$

$y = 41.66(x - x^4)$

Subdomain collocation method:

$$\int_0^1 R dx = 0$$

Sub R value in above eqn

$$\int_0^1 (-12a_1x^2 + 500x^2)dx = 0$$

$-12a_1[x^3/3]_0^1 + 500[x^3/3]_0^1 = 0$

$-12a_1/3[1-0]+500/3[1-0] = 0$

$-12a_1/3 + 500/3 = 0$

$-12a_1 = -500$

$a_1 = 41.66$

$y = 41.66(x - x^4)$

Least square method:

$$I = \int_0^1 R^2 dx = 0$$

$$\int_0^1 (-12a_1x^2 + 500x^2)^2 dx = 0$$

$$\int_0^1 [144a_1{}^2x^4 + 250000x^4 - 12000a_1x^4]dx = 0$$

$$\left[144a_1{}^2\frac{x^5}{5} + 250000\frac{x^5}{5} - 12000a_1\frac{x^5}{5}\right]_0^1 = 0$$

$$\frac{144a_1{}^2}{5} + \frac{250000}{5} - \frac{12000a_1}{5} = 0$$

$a_1 = 41.66$

$y = 41.66(x - x^4)$

Galerkin's method:

$$\int_0^1 w_i R dx = 0$$

$y = w_i = a_1(x - x^4)$

$$\int_0^1 a_1(x - x^4)(-12a_1x^2 + 500x^2)^2 dx = 0$$

$$a_1\int_0^1 (x - x^4)(-12a_1x^2 + 500x^2)dx = 0$$

$$a_1\int_0^1 (-12a_1x^3 + 500x^3 + 12a_1x^6 - 500x^6)dx$$

$$a_1\left[-12a_1\left[\frac{x^4}{4}\right]_0^1 + 500\left[\frac{x^4}{4}\right]_0^1 + 12a_1\left[\frac{x^7}{7}\right]_0^1 - 500\left[\frac{x^7}{7}\right]_0^1\right] = 0$$

$-3a_1+125+1.714a_1-71.428 = 0$

$-1.286a_1 = -53.572$

$a_1 = 41.66$

$y = 41.66(x - x^4)$

16. **The differential equation of a physical phenomenon is given by $\frac{d^2y}{dx^2}+500x^2=0$; , $0\le x\le 1$ by using the trial function, $y= a_1(x+x^3)+ a_2(x-x^5)$, calculate the values of the parameters a_1 and $a_2$by the following methods; Point collocation; (ii) subdomain collocation; (iii) least squares; (iv) galerkin. The boundary conditions are: y(0) = 0 , y(1) = 0. (Nov/Dec 2009)**

Given: $\frac{d^2y}{dx^2}+500x^2=0$ ------1

Boundary condition – y(0) = 0; y(1) =0

Trial function - $y= a_1(x+x^3)+ a_2(x-x^5)$

To find: (i) Point collocation;(ii) subdomain collocation; (iii) least squares; (iv) galerkin.

First boundary condition; x = 0, y = 0; $a_1 = 0$;

Second boundary condition; x= 1, y = 0, $a_1 = 0$

Residual R:

$$y = a_1(x-x^3) + a_2(x-x^5)$$

$$dy/dx = a_1(1-3x^2)+a_2(1-5x^4)$$

$$d^2y/dx^2 = a_1(-6x)+a_2(-20x^3)$$

$$d^2y/dx^2=-6a_1x-20a_2x^3$$

sub d^2y/dx^2 in eqn 1

$$R = -6a_1x-20a_2x^3+500x^2$$

Interval 0 to 1 is divided into two domain 0 to ½ and ½ to 1

Point collocation method:

$R = -6a_1x-20a_2x^3+500x^2 = 0$ ---3

Domain 1 Limit 0 – 1/2 = let it be x = 1/3

Put x = 1/3

$R = -6a_1(1/3)-20a_2(1/3)^3+500(1/3)^2=0$

$-2a_1-20a_2(1/27)+(500/9) = 0$

$2a_1+0.741a_2 = 55.55$

$a_1 + 0.3705a_2= 27.775$ ----4

Domain 2 limit 1/2 – 1 let us take x = 2/3

$R = -6a_1(3/3)-20a_2(2/3)^3+500(2/3)^2=0$

$-4a_1-20a_2(8/27)+(2000/9) = 0$

$-4a_1-5.925a_2 = -222.22$

$a_1 + 1.481a_2= 55.555$ ----5

solvingeqn 4 & 5

$a_2 = 25$; $a_1 = 18.53$

$y = 18.53(x-x^3) + 25 (x – x^5)$

Subdomain collocation method:

$$\int_0^{1/2} Rdx = 0$$

Domain 1 limits 0 - ½

Sub R value in above eqn

$$\int_0^{1/2}(-6a_1x \ -20a_2x^3 + 500x^2)dx = 0$$

Integrating the above equation

$$-6a_1\left[\frac{x^2}{2}\right]_0^{1/2} - 20a_2\left[\frac{x^4}{4}\right]_0^{1/2} + 500\left[\frac{x^3}{3}\right]_0^{1/2} = 0$$

$$\frac{-6a_1}{2}\left[\left(\frac{1}{2}\right)^2 - 0\right] - \frac{-20a_2}{4}\left[\left(\frac{1}{2}\right)^4 - 0\right] + \frac{500}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right] = 0$$

$$-\frac{6a_1}{8} - \frac{20a_2}{64} + \frac{500}{24} = 0$$

$-0.75a_1 - 0.3125a_2 + 20.83 = 0$

$a_1 + 0.4166a_2 = 27.773$ ----6

Domain 2 limits ½ - 1

$$\int_{1/2}^{1} Rdx = 0$$

Sub R value in above eqn

$$\int_{1/2}^{1}(-6a_1x \ -20a_2x^3 + 500x^2)dx = 0$$

$$-6a_1\left[\frac{x^2}{2}\right]_{1/2}^{1} - 20a_2\left[\frac{x^4}{4}\right]_{1/2}^{1} + 500\left[\frac{x^3}{3}\right]_{1/2}^{1} = 0$$

$$\frac{-6a_1}{2}\left[1-\left(\frac{1}{2}\right)^2\right] - \frac{20a_2}{4}\left[1-\left(\frac{1}{2}\right)^4\right] + \frac{500}{3}\left[1-\left(\frac{1}{2}\right)^3\right] = 0$$

$$-\frac{6a_1}{2}[0.75] - \frac{20a_2}{4}[0.9375] + \frac{500}{3}[0.875 = 0$$

$-2.25a_1 - 4.6875a_2 + 145.83 = 0$

$a_1 + 2.083a_2 = 64.813$ ----7

solvingeqn 6 & 7

$a_1 = 18.5$; $a_2 = 22.23$

$y = 18.50(x-x^3) + 22.53\ (x - x^5)$

<u>Least square method:</u>

Domain 1 limit 0 – 1/2

$$I = \int_0^{1/2} R^2 dx = 0$$

It can be written as

$$\frac{\partial I}{\partial a_1} = \int_0^{1/2} R \frac{\partial R}{\partial a_1} dx \text{ --------8}$$

$R = -6a_1x-20a_2x^3+500x^2 = 0$

$$\frac{\partial R}{\partial a_1} = -6x \text{ --------9}$$

Sub eqn 9 in eqn 8

$$\frac{\partial I}{\partial a_1} = \int_0^{1/2} (-6a_1x - 20a_2x^3 + 500x^2)\,(-6x)dx$$

$$\int_0^{1/2} (36a_1x^2 + 120a_2x^4 - 3000x^3)dx = 0$$

$$36a_1\left[\frac{x^3}{3}\right]_0^{\frac{1}{2}} + 120a_2\left[\frac{x^5}{5}\right]_0^{\frac{1}{2}} - 3000\left[\frac{x^4}{4}\right]_0^{\frac{1}{2}} = 0$$

$$\frac{36a_1}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right] + \frac{120a_2}{5}\left[\left(\frac{1}{2}\right)^5 - 0\right] - \frac{3000}{4}\left[\left(\frac{1}{2}\right)^4 - 0\right] = 0$$

$$\frac{12a_1}{8} + \frac{24a_2}{32} - \frac{750}{16} = 0$$

$$1.5a_1 + 0.75a_2 = 46.875$$

$a_1 + 0.5a_2 = 31.25$ ----10

Domain 2 limit 1/2 – 1

$$I = \int_{1/2}^{1} R^2 dx = 0$$

It can be written as

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^{1} R \frac{\partial R}{\partial a_1} dx \text{ --------11}$$

$R = -6a_1x-20a_2x^3+500x^2 = 0$

$$\frac{\partial R}{\partial a_2} = -20x^3 \text{ --------12}$$

Sub eqn 12 in eqn 11

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^{1} (\text{-}6a_1x - 20a_2x^3 + 500x^2)\,(-20x^3)dx$$

$$\int_{1/2}^{1} (120a_1x^4 + 400a_2x^6 - 10000x^5)dx = 0$$

$$120a_1\left[\frac{x^5}{5}\right]_{1/2}^{1} + 400a_2\left[\frac{x^7}{7}\right]_{1/2}^{1} - 10000\left[\frac{x^6}{6}\right]_{1/2}^{1} = 0$$

$$\frac{120a_1}{5}\left[(1)^5 - \left(\frac{1}{2}\right)^5\right] + \frac{400a_2}{7}\left[(1)^7 - \left(\frac{1}{2}\right)^7\right] - \frac{10000}{6}\left[(1)^6 - \left(\frac{1}{2}\right)^6\right] = 0$$

$$23.25a_1 + 56.695a_2 = 1640.626$$

$a_1 + 2.438a_2 = 70.564$ ----13

solvingeqn 10 & 13

$a_1 = 21.11$; $a_2 = 20.28$

$y = 21.11(x\text{-}x^3) + 20.28\,(x - x^5)$

<u>Galerkin's method:</u>

Domain 1 limit 0 – 1/2

$$\int_{0}^{1/2} w_i R dx = 0$$

$y = w_i = x - x^3$

Residual, $R = \text{-}6a_1x\text{-}20a_2x^3\text{+}500x^2 = 0$

$$\int_{0}^{1/2} (x - x^3)(\text{-}6a_1x - 20a_2x^3 + 500x^2)dx = 0$$

$$\int_{0}^{1/2} (-6a_1x^2 - 20a_2x^4 + 500x^3 + 6a_1x^4 + 20a_2x^6 - 500x^2)dx = 0$$

$$-6a_1\left[\frac{x^3}{3}\right]_0^{1/2} - 20a_2\left[\frac{x^5}{5}\right]_0^{1/2} + 500\left[\frac{x^4}{4}\right]_0^{1/2} + 6a_1\left[\frac{x^5}{5}\right]_0^{1/2} + 20a_2\left[\frac{x^7}{7}\right]_0^{1/2} - 500\left[\frac{x^6}{6}\right]_0^{1/2} = 0 \quad -$$

$$-2a_1(0.13) - 4a_2(0.031) + 125(0.063) + 1.2a_1(0.031) + 2.857a_2(0.008) - 83.33(0.015) = 0$$

$0.25a_1\text{-}0.125a_2\text{+}7.81\text{+}0.038a_1\text{+}0.022a_2\text{-}1.29 = 0$

$\text{-}0.2125a_1\text{-}0.1027a_2\text{+}6.5135 = 0$

$a_1 + 0.4832a_2 = 30.651$ ----14

Domain 2 limits ½ - 1

$$\int_{1/2}^{1} w_i R dx = 0$$

$y = w_2 = x - x^5$

Residual, $R = -6a_1x - 20a_2x^3 + 500x^2 = 0$

$$\int_{1/2}^{1} (x - x^5)(-6a_1x - 20a_2x^3 + 500x^2)dx = 0$$

$$\int_{1/2}^{1} (-6a_1x^2 - 20a_2x^4 + 500x^3 + 6a_1x^6 + 20a_2x^8 - 500x^7)dx = 0$$

-

$$-6a_1\left[\frac{x^3}{3}\right]_{1/2}^{1} - 20a_2\left[\frac{x^5}{5}\right]_{1/2}^{1} + 500\left[\frac{x^4}{4}\right]_{1/2}^{1} + 6a_1\left[\frac{x^7}{7}\right]_{1/2}^{1} + 20a_2\left[\frac{x^9}{9}\right]_{1/2}^{1} - 500\left[\frac{x^8}{8}\right]_{1/2}^{1} = 0$$

$1.75\, a_1 - 3.875\, a_2 + 117.187 + 0.850\, a_1 + 2.215\, a_2 - 62.255 = 0$

$-0.9a_1 - 1.659a_2 + 54.932 = 0$

$a_1 + 1.843a_2 = 61.035$ ----15

Solving eqn 14 & 15

$a_1 = 19.862$; $a_2 = 22.34$

$y = 19.862(x - x^3) + 22.34\,(x - x^5)$

17. The differential equation of a physical phenomenon is given by $\frac{d^2y}{dx^2}$-10x^2=5. Obtain two term galerkin solution by using the trial functions: $N_1(x)=x(x-1)$; $N_2(x)=x^2(x-1)$; $0 \leq x \leq 1$. boundary conditions are, y (0)=0; y (1)=0.

Given: $\frac{d^2y}{dx^2} - 10x^2 = 5$ - - 1

Trial functions, $N_1(x)=x(x-1)$; $N_2(x)=x^2(x-1)$

$y = a_1x(x-1) + a_2x^2(x-1)$ - - 2

BC - - $y(0) = 0$; $y(1) = 0$

To find: approximate solution using Galerkin's method.

Solution:

Trial function

$$y = a_1x(x-1) + a_2x^2(x-1)$$

when $x = 0$; $y=0$

$x=1$; $y=0$

Residual R

$R = y = a_1x(x-1) + a_2x^2(x-1)$

$y = a_1(x^2-x)+a_2(x^3-x^2)$

$dy/dx = a_1(2x-1)+a_2(3x^2-2x)$

$d^2y/dx^2 = a_1(2)+a_2(6x-2)$

$R = 2a_1 + 6a_2x - 2a_2 - 10x^2 - 5$-----3

Using Galerkin method to find the solution for the problem.

$$\int_0^1 w_i R dx = 0$$

Here $w_1 = x(x-1)$

$w_2 = x^2(x-1)$

$$\int_0^1 x(x-1)(2a_1 + 6a_2 x - 2a_2 - 10x^2 - 5)dx = 0 \text{ ---4}$$

$$\int_0^1 x^2(x-1)(2a_1 + 6a_2 x - 2a_2 - 10x^2 - 5)dx = 0 \text{ ----5}$$

Integrating the above equation 4 and 5 we get

$$\int_0^1 x(x-1)(2a_1+6a_2x-2a_2-10x^2-5)dx=0$$

$$\int_0^1 (x^2-x)(2a_1+6a_2x-2a_2-10x^2-5)dx=0$$

$$\int_0^1 (2a_1x^2+6a_2x^3-2a_2x^2-10x^4-5x^2-2a_1x-6a_2x^2+2a_2x+10x^3+5x)dx=0$$

$$\left[(2a_1\frac{x^3}{3}+6a_2\frac{x^4}{4}-2a_2\frac{x^3}{3}-10\frac{x^5}{5}-5\frac{x^3}{3}-2a_1\frac{x^2}{2}-6a_2\frac{x^3}{3}+2a_2\frac{x^2}{2}x+10\frac{x^4}{4}+5\frac{x^2}{2}\right]_0^1=0$$

$$\left[(2a_1\frac{1^3}{3}+6a_2\frac{1^4}{4}-2a_2\frac{1^3}{3}-10\frac{1^5}{5}-5\frac{1^3}{3}-2a_1\frac{1^2}{2}-6a_2\frac{1^3}{3}+2a_2\frac{1^2}{2}x+10\frac{1^4}{4}+5\frac{1^2}{2}\right]=0$$

$$0.66a_1+1.5a_2-0.66a_2-2-1.66-a_1-2a_2+a_2+2.5+2.5=0$$

$$-0.33a_1-0.166a_2+1.334=0$$

$a_1+0.5a_2 = 4$ ----6

$$\int_0^1 x^2(x-1)(2a_1+6a_2x-2a_2-10x^2-5)dx=0$$

$$\int_0^1 (x^3-x^2)(2a_1+6a_2x-2a_2-10x^2-5)dx=0$$

$$\int_0^1 (2a_1x^3+6a_2x^4-2a_2x^3-10x^5-5x^3-2a_1x^2-6a_2x^3+2a_2x^2+10x^4+5x^2)dx=0$$

$$\left[(2a_1\frac{x^4}{4}+6a_2\frac{x^5}{5}-2a_2\frac{x^4}{4}-10\frac{x^6}{6}-5\frac{x^4}{4}-2a_1\frac{x^3}{3}-6a_2\frac{x^4}{4}+2a_2\frac{x^3}{3}x+10\frac{x^5}{5}+5\frac{x^3}{3}\right]_0^1=0$$

$$\left[(2a_1\frac{1^4}{4}+6a_2\frac{1^5}{5}-2a_2\frac{1^4}{4}-10\frac{1^6}{6}-5\frac{1^4}{4}-2a_1\frac{1^3}{3}-6a_2\frac{1^4}{4}+2a_2\frac{1^3}{3}x+10\frac{1^5}{5}+5\frac{1^3}{3}\right]=0$$

$$0.5a_1+1.2a_2-0.5a_2-1.66-1.25-0.66a_1-1.5a_2+0.66a_2+2+1.66=0$$

$$0.166a_1+0.133a_2+0.75=0$$

$a_1+0.8012a_2 = 4.518$-----7

solvingeqn 6 & 7

$a_2 = 1.719$; $a_1 = 3.140$

$y = 1.719x^3 + 1.421x^2 - 3.140\,x$

18. The differential equation of a physical phenomenon is given by $\frac{d^2y}{dx^2}+y=4x$, $0\leq x\leq 1$.

19. The boundary conditions are: y(0) = 0 , : y(1) = 0. Obtain one term approximate solution by using galerkin's method of weighted residuals.(May/June 2014).

Given: $\frac{d^2y}{dx^2}$+y=4x --------1

BC, y(0) = 0; y(1) = 1

To find: approximate solution using Galerkin's method

Solution:

Assume trial function;

$$y = a_1x(x-1)+x$$

when x = 0; y = 0

x = 1; y =1

The give equation (trial function) satisfies the boundary condition.

R = $y = a_1x(x-1) + x$ --- 2

$y = a_1(x^2-x) + x$

$dy/dx = a_1(2x – 1) + 1$

$d^2y/dx^2 = 2a_1$

sub d^2y/dx^2 in eqn 1

$2a_1 + y = 4x$

Sub y value

$2a_1 + a_1x(x-1)+x = 4x$

$R = 2a_1 + a_1x(x – 1) + x – 4x$

Using Galerkin method to find the solution for the problem.

$$\int_0^1 w_i R dx = 0 \quad \text{- - 3}$$

$$\int_0^1 x(x-1)(2a_1 + a_1x(x-1) + x - 4x)dx = 0$$

$$\int_0^1 x(x-1)(2a_1 + a_1x^2 - a_1x + x - 4x)dx = 0$$

$$\int_0^1 (x^2 - x)(2a_1 + a_1x^2 - a_1x + x - 4x)dx = 0$$

$$\int_0^1 (2a_1x^2 + a_1x^4 - a_1x^3 + x^3 - 4x^3 - 2a_1x - a_1x^3 + a_1x^2 - x^2 + 4x^2)dx = 0$$

$$\left[(2a_1\frac{x^3}{3} + a_1\frac{x^5}{5} - a_1\frac{x^4}{4} + \frac{x^4}{4} - 4\frac{x^4}{4} - 2a_1\frac{x^2}{2} - a_1\frac{x^4}{4} + a_1\frac{x^3}{3} - \frac{x^3}{3} + 4\frac{x^3}{3})\right]_0^1 = 0$$

$$\left[\frac{2a_1}{3} + \frac{a_1}{5} - \frac{a_1}{4} + \frac{1}{4} - \frac{4}{4} - \frac{2a_1}{2} - \frac{a_1}{4} + \frac{a_1}{3} - \frac{1}{3} + \frac{4}{3}\right] = 0$$

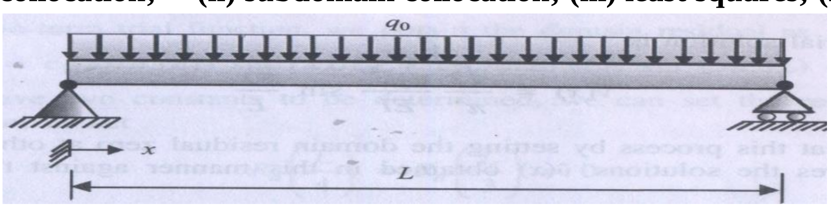
$$0.66a_1 + 0.2a_1 - 0.25a_1 - 0.75 - a_1 - 0.25a_1 + 0.33a_1 + 1 = 0$$

$$-0.301a_1 = -0.25$$

$a_1 = 0.830$

$y = 0.830x^2 + 0.17x$

20. Find the deflection at the centre of a simply supported beam of span length 'l' subjected to uniformly distributed load through out its length as shown in fig. using (i) Point collocation; (ii) subdomain collocation; (iii) least squares; (iv) galerkin.



Solution:

EI $d^4y/dx^4 - \omega = 0$, $0 \le x \le l$ - - - 1

BC – y = 0 at x = 0 & x = l when y is the deflection

EI $d^4y/dx^4 = 0$ at x =0 and x=l

Where, EI d^4y/dx^4 = M (bending moment)

I = moment of inertia of the beam

E = young's modulus.

Let us assume trial function

$y = a \sin(\pi x/l)$ - -2

$d^4y/dx^4 = a\pi^4/l^4 \sin(\pi x/l)$ – 3

subeqn 3 in 1

EI $(a\pi^4/l^4 \sin(\pi x/l)) - \omega = 0$

The above equation satisfies the boundary condition

$$\frac{dy}{dx} = \frac{a\pi}{l}.\cos\frac{\pi x}{l}$$
$$\frac{d^2y}{dx^2} = -\frac{a\pi^2}{l^2}.\sin\frac{\pi x}{l}$$
$$\frac{d^3y}{dx^3} = -\frac{a\pi^3}{l^3}.\cos\frac{\pi x}{l}$$
$$\frac{d^4y}{dx^4} = \frac{a\pi^4}{l^4}.\sin\frac{\pi x}{l}$$

Sub eqn 3 in eqn 1

$$EI\left(\frac{a\pi^4}{l^4}.\sin\frac{\pi x}{l}\right) - \omega = 0$$

$$\text{Re}\,sidua, R = EI\,\frac{a\pi^4}{l^4}.\sin\frac{\pi x}{l} - \omega$$

<u>Point collocation method:</u>

R = EI $(a\pi^4/l^4 \sin(\pi x/l)) - \omega = 0$

EI $(a\pi^4/l^4 \sin(\pi x/l)) = \omega$

Take $x = l/2$

EI $(a\pi^4/l^4 \sin(\pi x/l)) = \omega$

$$EI\,\frac{a\pi^4}{l^4}\sin\frac{\pi}{l}\left(\frac{l}{2}\right) = \omega$$

$$EI\,\frac{a\pi^4}{l^4} = \omega$$

$$a = \frac{\omega l^4}{\pi^4 EI}$$

Substitute ‘a’ value in the trial function

$y = \omega l^4/\pi^4 EI \sin(\pi x/l)$

<u>Sub domain collocation method:</u>

$$\int_0^1 R\,dx = 0$$

R = EI $(a\pi^4/l^4 \sin(\pi x/l)) - \omega = 0$

$$\int_0^l \left(aEI\,\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - \omega \right) dx = 0$$

$$\left[aEI\,\frac{\pi^4}{l^4}\left(\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}} \right) - \omega x \right]_0^l = 0$$

$$\left[aEI\,\frac{\pi^4}{l^4}\left(-\cos\frac{\pi x}{l} \right)\left(\frac{l}{\pi} \right) - \omega x \right]_0^l = 0$$

$$-aEI\,\frac{\pi^3}{l^3}(\cos\pi - \cos 0) - \omega l = 0$$

$$-aEI\,\frac{\pi^3}{l^3}(-1-1) = \omega l$$

$a = \omega l^4/2\pi^3EI$

$a = \omega l^4/62EI$

$y = \omega l^4/62EI \sin(\pi x/l)$

Least square method

$$I = \int_0^1 R^2\,dx = 0$$

$$I = \int_0^l \left(aEI\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - \omega \right)^2 dx$$

$$= \int_0^l \left(a^2E^2I^2\frac{\pi^8}{l^8}\sin^2\frac{\pi x}{l} + \omega^2 - 2aEI\omega\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} \right) dx$$

$$= \int_0^l \left(a^2E^2I^2\frac{\pi^8}{l^8}\left(\frac{1-\cos\left(\frac{2\pi x}{l}\right)}{2} \right) + \omega^2 - 2aEI\omega\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} \right) dx$$

$$= \left[a^2E^2I^2\frac{\pi^8}{l^8}\left(\frac{1}{2}x - \sin\frac{2\pi x}{l}\left(\frac{l}{2\pi} \right) \right) + \omega^2 x - 2aEI\omega\frac{\pi^4}{l^4}\left(-\cos\frac{\pi x}{l}\left(\frac{l}{\pi} \right) \right) \right]_0^l$$

$$= \left[a^2E^2I^2\frac{\pi^8}{l^8}\left(\frac{1}{2}l - \frac{l}{2\pi}(\sin 2\pi - \sin 0) \right) + \omega^2 l + 2aEI\omega\frac{\pi^4}{l^4}\frac{l}{\pi}\left(\cos\pi - \cos 0 \right) \right]$$

$$I = a^2E^2I^2\frac{\pi^8}{l^8}\frac{l}{2} + \omega^2 l + 2aEI\omega\frac{\pi^3}{l^3}(-1-1)$$

$$I = \frac{a^2E^2I^2\pi^8}{2l^7} + \omega^2 l - 4aEI\omega\frac{\pi^3}{l^3}$$

Now, $\frac{\partial I}{\partial a} = 0$

$$\frac{2aE^2I^2\pi^8}{2l^7} = 4EI\omega\frac{\pi^3}{l^3}$$

a = $4\omega l^4/\pi^5 EI$

y = $4\omega l^4/\pi^5 EI \sin(\pi x/l)$

Galerkin's method:

$$\int_0^1 w_i R\,dx = 0$$

$w_i = y = a\sin(\pi x/l)$

$$\int_0^l \left[\left(a\sin\frac{\pi x}{l} \right)\left(aEI\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - \omega \right) \right] dx = 0$$

$$\int_0^l \left[a^2EI\frac{\pi^4}{l^4}\sin^2\frac{\pi x}{l} - a\sin\frac{\pi x}{l} \right] dx = 0$$

$$\int_0^l \left[a^2 EI \frac{\pi^4}{l^4} \left[\frac{1}{2} \left(1 - \cos \frac{2\pi x}{l} \right) \right] - a\omega \sin \frac{\pi x}{l} \right] dx = 0$$

$$\left[a^2 EI \frac{\pi^4}{l^4} \left(\frac{1}{2} \left[x - \left(\frac{1}{2\pi} \right) \sin \frac{2\pi x}{l} \right] \right) + a\omega \frac{1}{\pi} \left(\cos \frac{\pi x}{l} \right) \right]_0^l = 0$$

$$a^2 EI \frac{\pi^4}{l^4} \left(\frac{l}{2} \right) - 2a\omega \left(\frac{l}{\pi} \right) = 0$$

$$a = 4\omega l^4/\pi^5 EI$$

$$y = 4\omega l^4/\pi^5 EI \sin(\pi x/l)$$

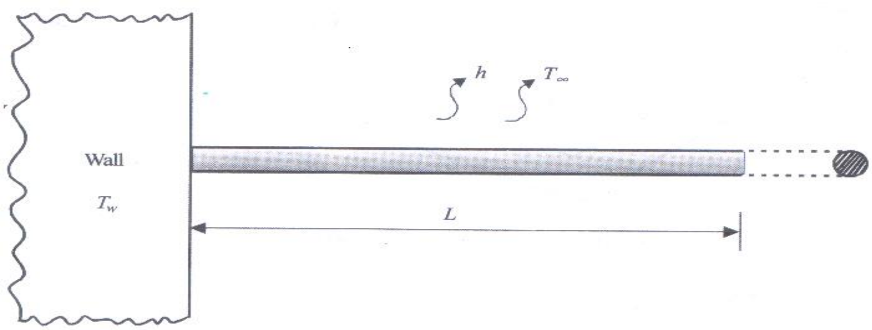
21. Consider a 1mm diameter, 50 mm long aluminum pin-fin as shown fig. used to enhance the heat transfer from a surface wall maintained at 300˚C. the governing differential equation and the boundary conditions are given by

$$k\frac{d^2T}{dx^2} = \frac{p\,h}{A}(T - T_\infty)$$

$$T(0) = T_w = 300^\circ C$$

$$\frac{d^2T}{dx^2} L = 0 \text{ (insulated tip)}$$

Where, k=thermal conductivity, p= perimeter, A= cross-sectional area, h= convective heat transfer coefficient, T_w=wall temperature, T_∞=ambient temperature.Let, k=200W/m˚C for aluminum, h=20 W/m^2˚C, T_∞=30˚C.estimate the temperature distribution in the fin using the galerkin weighted residual method.



Given:

Diameter, d=1mm=1x10^{-3}

Length , l= 50mm= 50x10^{-3}m

Wall temperature, T_w=300˚C

Governing different equation, $K\frac{d^2T}{dx^2} = \frac{p\,h}{A}(T - T_\infty)$

$$T(0) = T_w = 300^\circ C$$

$$\frac{dT}{dx}(L) = 0$$

Thermal conductivity, k=200w/m˚C.

Heat transfer co-efficient h = 20 W/m^{20}C

Ambient temperature, T_∞ = 30^{0}C

To find: Temperature distribution using Galerkin's method.

Solution:

Assume a trail solution. Let.

$$T(x)=a_0+a_1x+a_2x^2$$

The boundary conditions are, $T(0)=T_w=300^\circ C$

$$\frac{dT}{dx}(L)=0$$

From equation (a), x=0, T=300˚C

Applying these values in equation (1),

$$300=a_0$$

$$a_0=300$$

From equation (b), x=L, $\frac{dT}{dx}=0$

Differential equation (1), $\frac{dT}{dx}=a_1+a_2.2x$ -------------2

$$0=a_1+a_2(2L)$$

$$a_1 = -2La_2$$

Substitute a_0 and a_1 values in equation (1)

$$T(x)=a_0+a_1x+a_2x^2$$

$$T(x) = 300 + (-2a_2L)x+a_2x^2$$

$$T(x) = 300 + a_2(x^2-2Lx) \text{------- } 3$$

$$\text{w.k.t, } kd^2T/dx^2 = Ph/A\ (T - T_\infty)$$

$$200\ d^2T/dx^2 = \pi(1*10^{-3})*20/(\pi/4)(1*10^{-3})^2\ \ (T-30) \qquad [\text{here } p=\pi D]$$

$$d^2T/dx^2 = 400\ (T-30) \text{ - - - - - } 4$$

substitute T value from eqn 3

$$d^2T/dx^2 = 400\ (300 + a_2(x^2-2Lx)-30)$$

$$=400[270+a_2(x^2-2Lx)] \text{ - - - } 5$$

From eqn 2, $dT/dx = a_1+a_2(2x)$

$d^2T/dx^2 = 2a_2$ ------ 6

substitute d^2T/dx^2 value in equation 5

$2a_2= 400[270+a_2(x^2-2Lx)]$

$2a_2- 400[270+a_2(x^2-2Lx)] = 0$

Take residual, $R=2a_2-400[270+a_2(x^2-2Lx)$

$W(x) = x^2-2Lx$

$$\int_0^l (x^2-2Lx)Rdx = 0$$
$$\int_0^L (x^2-2Lx)[2a_2-400(270+a_2(x^2-2Lx))]dx = 0$$
$$\int_0^L (x^2-2Lx)[2a_2-108000-400a_2x^2+800a_2Lx]dx = 0$$
$$\int_0^L [2a_2x^2-108000x^2-400a_2x^4+800a_2Lx^3-4a_2Lx+216000Lx+800a_2Lx^3-1600a_2L^2x^2]dx = 0$$
$$\left[2a_2\frac{x^3}{3}-108000\frac{x^3}{3}-400a_2\frac{x^5}{5}+800a_2L\frac{x^4}{4}-4a_2L\frac{x^2}{2}+216000L\frac{x^2}{2}+800a_2L\frac{x^4}{4}-1600a_2L^2\frac{x^3}{3}\right]_0^L = 0$$
$$\left[2a_2\frac{L^3}{3}-108000\frac{L^3}{3}-400a_2\frac{L^5}{5}+800a_2L\frac{L^4}{4}-4a_2L\frac{L^2}{2}+216000L\frac{L^2}{2}+800a_2L\frac{L^4}{4}-1600a_2L^2\frac{L^3}{3}\right] = 0$$
$$\left[2a_2\frac{L^3}{3}-108000\frac{L^3}{3}-400a_2\frac{L^5}{5}+800a_2\frac{L^5}{4}-4a_2\frac{L^3}{2}+216000\frac{L^3}{2}+800a_2\frac{L^5}{4}-1600a_2\frac{L^5}{3}\right] = 0$$
$$\left[\frac{2a_2}{3}-\frac{108000}{3}-\frac{400a_2L^2}{5}+\frac{800a_2L^2}{4}-\frac{4a_2}{2}+\frac{216000}{2}+\frac{800a_2L^2}{4}-\frac{1600a_2L^2}{3}\right] = 0$$
$$a_2[0.667-80L^2+200L^2-2+200L^2-533.33L^2] = -72000$$
$$L = 50*10^{-3}m(given)$$
$$a_2[0.667-0.2+0.5-2+0.5-1.33] = -72000$$
$$a_2 = 38572.8$$

Galerkin solution, $T(x)=300+38572.80(x^2-2Lx)$

22. Write short notes on Variational Formulation and Ritz Technique.

VARIATIONAL FORMULATION:

Variational (Weak) Form of the weighted residual statement:

The general weighted residual statement is

$\int$ w R dx =0

In this vibrational method, integration is carried out by parts. It reduces the continuity requirement on the trail function assumed in the solution. So it is referred to as the weak form. In this method, it is possible to have a wider choice of trial function.

Characteristics of weighted residual statement:

- Weighted residual statement can be developed for any form of differential equations like linear non-linear, ordinary, partial, etc.
- The weighted residual statement is developed only for differential equation and it is not suitable for boundary conditions.
- The trial solution should satisfy all the boundary conditions and it should be differentiable as many times as needed in the original differential equation.

RITZ TECHNIQUES:

Rayleigh-Ritz Method:

Rayleigh-Ritz Method is a vibrational approach and it is an integral approach method which will be very much helpful in solving complex structural problems, found in FEA. This method is mostly used for solving solid mechanics problems. This method is possible only if a suitable functional is available. Otherwise, Galerkin's method of weighted residual is used.

Rayleigh-Ritz Method is a vibrational method because that makes use of vibrational principle, such as the principles of virtual work and the principle of minimum potential energy in solid and structural mechanics to determine the approximate solutions of the problem.

Total potential energy of the structure is given by

Π = U-H

Where,

Π – Total potential energy

H – External potential energy (or) work done by external forces

U – Internal potential energy (or) strain energy

In Rayleigh ritz method, the approximating functions must satisfy the boundary conditions and should be easy to use.

Polynomials are used some times, otherwise sine and cosine functions will be used as approximating functions.

For representing exact functions, the following two terms are used.

$y=a_0+ a_1x+a_2x^2+a_3x^3+\ldots.$

$y= a_1\sin\left(\frac{\Pi x}{l}\right) + a_2\sin\left(\frac{3\Pi x}{l}\right) +\ldots..$

Where,

a_0, a_1, a_2 ---- unknown Riz parameter.

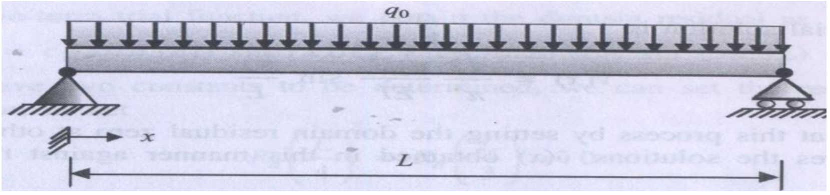
The following conditions must satisfy the approximating function:

- It should satisfy the geometric boundary conditions.
- The function must have atleast one Ritz parameters.

Advantages of the weak form

- Order of the differential equation becomes half of that in the original equation.
- Hence continuity requirements on the assumed solution is reduced.
- Lower order polynomial can be assumed for the approximate solution.
- The Natural Boundary condition becomes embedded in the weak form
- Hence the trial solution needs to satisfy only the essential boundary condition

23. A beam AB span 'l' simply supported at ends. And carrying a concentrated load W at the centre 'C' as shown in fig. determine the deflection at mid span by using Rayleigh -Ritz method and compare with exact solutions.



Solution: we know that,

$$\pi=U-H$$

U=strain energy

H=work done external force

The strain energy u of beam due to loading is given by,

$$U=\frac{EI}{2}\int_0^1\left(\frac{d^2u}{dx^2}\right)^2dx$$

from equation in previous example, we know that,

$$U=\frac{E\,\pi I}{4\,l^3}[a_1^2+8la_2^2]$$

Work done by external force, H=W y_{max}

Defelction, $y=a_1\sin\frac{\pi x}{l}+a_2\sin\frac{3\pi x}{l}$

In the span, deflection is maximum at $x=\frac{l}{2}$

$$y_{max}=a_1\sin\frac{\frac{\pi l}{2}}{l}+a_2\sin\frac{\frac{3\pi l}{2}}{l}$$

$$=a_1\sin\frac{\pi}{2}+a_2\sin\sin\frac{3\pi}{2}$$

$y_{max}=a_1-a_2$

$H=W(a_1-a_2)$

Substitute U and H values in equation

$$\pi=\frac{E\,\pi I}{4\,l^3}[a_1^2+8la_2^2]-W(a_1-a_2)$$

For stationary value π, the following condition must be satisfied

$$\frac{\partial\pi}{\partial a_1}=0 \quad\text{and}\quad \frac{\partial\pi}{\partial a_2}=0$$

$$\frac{\partial\pi}{\partial a_1}=\frac{EI\pi^4}{4l^3}(2a_1)-W=0$$

$$\frac{EI\pi^4}{2l^3}(a_1)-W=0$$

$$\frac{EI\pi^4}{2l^3}(a_1)=W$$

$$a_1=\frac{2l^3}{EI\pi^4}(W)$$

Similarly,

$$\frac{\partial\pi}{\partial a_2}=\frac{EI\pi^4}{4l^3}(162a_2)+W=0$$

$$\frac{EI\pi^4}{4l^3}(162a_2)=-W$$

$$a_2=\frac{4l^3}{162EI\pi^4}(-W)$$

$$a_2=\frac{2l^3}{81EI\pi^4}(-W)$$

$$y_{Max}=a_1-a_2$$

$$a_1=\frac{2\,W\,l^3}{EI\,\pi^4}$$

$$a_2=\frac{-2\,W\,l^{\,3}}{8\,lEI\,\pi^4}$$

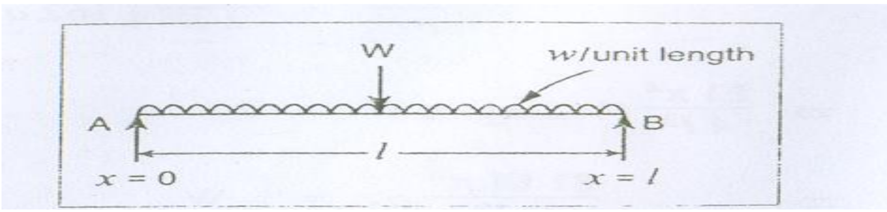
$$Y_{max}=\frac{Wl^3}{48.1\,EI}$$

We know that, simply supported beam subjected to point load at center, maximum deflection is ,

$$Y_{max}=\frac{Wl^3}{48\,EI}$$

From equation, we know that. Exact solution and solution by using Rayleigh-ritz method are almost same. In order to get accurate result, more terms in fourier series should be taken.

24. A simply supported beam subjected to uniformly distributed load over entire span and it is subjected to a point load at the center of the span. Calculate the bending moment and deflection at midspan by using rayleigh-ritz method and compare with exact solution. (May/June 2013, Nov/Dec 2014).



To find :1. Deflection and bending moment at midspan.

2. compare with exact solution.

Solution: Defelction, $y=a_1\sin\frac{\pi x}{l}+a_2\sin\frac{3\pi x}{l}$

Total potential energy of the beam is given by,

$$\pi=U-H$$

The strain energy U of the beam due to bending is given by,

$$U=\frac{EI}{2}\int_0^l\left(\frac{d^2y}{dx^2}\right)^2dx$$

From equation in

$$U=\frac{E\,I\pi^4}{4\,l^3}[a_1^2+8la_2^2]$$

Work done by external force, $H=\int_0^1\omega\ y\ dx+W\ y_{max}$

From equation $\int_0^1\omega\ y\ dx=\frac{2\,\omega\,l}{\pi}(a_1+\frac{a_2}{3})$

$$y_{max}=a_1\sin\frac{\frac{\pi l}{2}}{l}+a_2\sin\frac{\frac{3\pi l}{2}}{l}$$

$$=a_1\sin\frac{\pi}{2}+a_2\sin\sin\frac{3\pi}{2}$$

$y_{max}=a_1-a_2$

substituting the above value in equation

$$H=\frac{2\,\omega\,l}{\pi}(a_1+\frac{a_2}{3})+W(a_1-a_2)$$

substituting the U & H value in equation

$$\pi=\frac{EI\pi^4}{4l^3}\left[a_1^{\,2}+81a_2^{\,2}\right]-\left[\frac{2\omega l}{\pi}\left(a_1+\frac{a_2}{3}\right)+W(a_1-a_2\right]$$

$$\pi=\frac{EI\pi^4}{4l^3}\left[a_1^{\,2}+81a_2^{\,2}\right]-\frac{2\omega l}{\pi}\left(a_1+\frac{a_2}{3}\right)-W(a_1-a_2)$$

For stationary value π, the following condition must be satisfied

$$\frac{\partial\pi}{\partial a_1}=0 \quad \text{and} \quad \frac{\partial\pi}{\partial a_2}=0$$

$$\frac{\partial\pi}{\partial a_1}=\frac{EI\pi^4}{4l^3}(2a_1)-\frac{2\omega l}{\pi}-W=0$$

$$\frac{EI\pi^4}{2l^3}(a_1)-\frac{2\omega l}{\pi}-W=0$$

$$\frac{EI\pi^4}{2l^3}(a_1)=\frac{2\omega l}{\pi}+W$$

$$a_1=\frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi}+W\right)$$

Similarly,

$$\frac{\partial\pi}{\partial a_2}=\frac{EI\pi^4}{4l^3}(162a_2)-\frac{2\omega l}{\pi}\left(\frac{1}{3}\right)+W=0$$

$$\frac{EI\pi^4}{4l^3}(162a_2)=\frac{2\omega l}{3\pi}-W$$

$$a_2=\frac{4l^3}{162EI\pi^4}\left(\frac{2\omega l}{3\pi}-W\right)$$

$$a_2=\frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi}-W\right)$$

$y_{max}= a_1-a_2$

Substitute a_1 and a_2 value in the above equation

$$y_{max}=\frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi}+W\right) - \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi}-W\right)$$

$$y_{max}=\frac{4\omega l^4}{EI\pi^5}+\frac{2Wl^3}{EI\pi^4} - \frac{4l^4}{243EI\pi^5}+\frac{2Wl^3}{81EI\pi^4}$$

$$y_{max}=\frac{4\omega l^4}{EI\pi^5}\left(1-\frac{1}{243}\right)+\frac{2Wl^3}{EI\pi^4}\left(1+\frac{1}{81}\right)$$

$$y_{max}=\frac{3.98\omega l^4}{EI\pi^5}+\frac{2.02Wl^3}{EI\pi^4}$$

$$y_{max}=\left[0.0130\frac{\omega l^4}{EI\pi^5}+0.0207\frac{Wl^3}{EI\pi^4}\right]$$

W.k.t, simply supported beam subjected to uniformly distributed load, max deflection is,

$$y_{max}=\left[\frac{5\omega l^4}{384EI}\right]$$

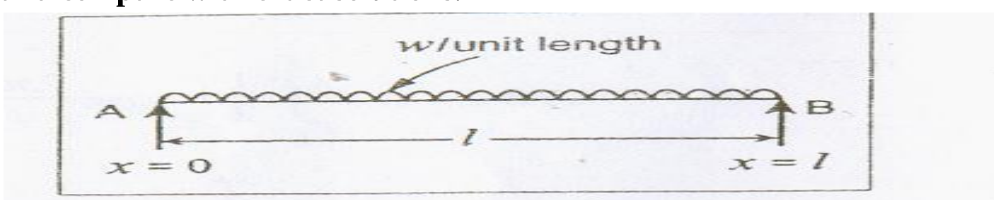
Simply supported beam subjected to point load at centre, max deflection is

$$y_{max}=\left[\frac{Wl^4}{48EI}\right]$$

Total deflection

$$y_{max}=\left[0.0130\frac{\omega l^4}{EI\pi^5}+0.0207\frac{Wl^3}{EI\pi^4}\right]$$

25. A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at midspan by using rayleigh-ritz method and compare with exact solutions.



To find: we know that, for simply supported beam, the fourier series,

$Y=\sum_{n=1.3}^{\infty} a\, sin\frac{\pi n x}{l}$ is the approximating function.

To make this series more simple let us consider only two terms.

Deflection, $y=a_1 sin\frac{\pi x}{l}+a_2 sin\frac{3\pi x}{l}$

Where, a_1, a_2 are Ritz parameters.

We know that,

Total potential energy of the beam, π=U-H

H=work done external force

The strain energy, U, of the beam due to bending is given by,

$U=\frac{EI}{2}\int_0^1(\frac{d^2y}{dx^2})^2dx$

$\frac{dy}{dx}=a_1 cos\frac{\pi x}{l}(\frac{\pi}{l})+a_2 cos\frac{3\pi x}{l}(\frac{3\pi}{l})$

$\frac{dy}{dx}=\frac{a_1\pi}{l}\cos\frac{\pi x}{l}+\frac{a_2\,3\pi}{l}\cos\frac{3\pi x}{l}$

$$\frac{d^2y}{dx^2}=-\frac{a_1\pi}{l}\sin\frac{\pi x}{l}*\frac{\pi}{l}-\frac{a_2 3\pi}{l}\sin\frac{3\pi x}{l}*\frac{3\pi}{l}$$

$$=-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l}-\frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

Substitute $\frac{d^2y}{dx^2}$ value in equation

$$U=\frac{EI}{2}\int_0^l\left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l}-\frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2dx$$

$$=\frac{EI}{2}\int_0^l\left[\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l}+\frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2dx$$

$$=\frac{EI}{2}\int_0^l\left[\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l}+\frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2dx$$

$$=\frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l\left[a_1\sin\frac{\pi x}{l}+9a_2\sin\frac{3\pi x}{l}\right]^2dx$$

$$=\frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l\left[a_1^2\sin^2\frac{\pi x}{l}+81a_2^2\sin^2\frac{3\pi x}{l}+18a_1a_2\sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}\right]dx$$

$$=\frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l\left[a_1^2\left(\frac{1-\cos\frac{2\pi x}{l}}{2}\right)+81a_2^2\left(\frac{1-\cos\frac{6\pi x}{l}}{2}\right)+18a_1a_2\left(\frac{\cos\frac{2\pi x}{l}-\cos\frac{4\pi x}{l}}{2}\right)\right]dx$$

$$=\frac{EI}{4}\frac{\pi^4}{l^4}\int_0^l\left[a_1^2\left(1-\cos\frac{2\pi x}{l}\right)+81a_2^2\left(1-\cos\frac{6\pi x}{l}\right)+18a_1a_2\left(\cos\frac{2\pi x}{l}-\cos\frac{4\pi x}{l}\right)\right]dx$$

$$=\frac{EI}{4}\frac{\pi^4}{l^4}\left[a_1^2\left(x-\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)\right)+81a_2^2\left(x-\left(\frac{\sin\frac{6\pi x}{l}}{\frac{6\pi}{l}}\right)\right)+18a_1a_2\left(\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)-\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)\right)\right]_0^1$$

$$=\frac{EI}{4}\frac{\pi^4}{l^4}\left[a_1^2\frac{l}{2}+81a_2^2\frac{l}{2}+18a_1a_2(0)\right]$$

$$U=\frac{EI}{4}\frac{\pi^4}{l^3}\left[a_1^2+81a_2^2\right]$$

Work done by external force

$$H=\int_0^l\omega y\,dx$$

$$H = \int_0^l \left[\omega a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \right] dx$$

$$= \omega \int_0^l \left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \right) dx$$

$$= \omega \left(a_1 \left(\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}} \right) + a_2 \sin \left(\frac{-\cos\frac{3\pi x}{l}}{\frac{3\pi}{l}} \right) \right)_0^l$$

$$= \omega \left(\frac{2a_1 l}{\pi} + \frac{2a_2 l}{3\pi} \right)$$

$$H = \frac{2\omega l}{\pi} \left(a_1 + \frac{a_2}{\pi} \right)$$

The final equation be

$$\pi = U - H$$

$$\pi = \frac{EI}{4} \frac{\pi^4}{l^3} \left[a_1^2 + 81 a_2^2 \right] - \frac{2\omega l}{\pi} \left(a_1 + \frac{a_2}{\pi} \right)$$

For stationary value π, the following condition must be satisfied

$$\frac{\partial \pi}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial a_2} = 0$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI\pi^4}{4l^3} (2a_1) - \frac{2\omega l}{\pi} = 0$$

$$\frac{EI\pi^4}{2l^3} (a_1) - \frac{2\omega l}{\pi} = 0$$

$$\frac{EI\pi^4}{2l^3} (a_1) = \frac{2\omega l}{\pi}$$

$$a_1 = \frac{2l^3}{EI\pi^4} \left(\frac{2\omega l}{\pi} \right)$$

Similarly,

$$\frac{\partial \pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2\omega l}{\pi}\left(\frac{1}{3}\right) = 0$$

$$\frac{EI\pi^4}{4l^3}(162a_2) = \frac{2\omega l}{3\pi}$$

$$a_2 = \frac{4l^3}{162EI\pi^4}\left(\frac{2\omega l}{3\pi}\right)$$

$$a_2 = \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi}\right)$$

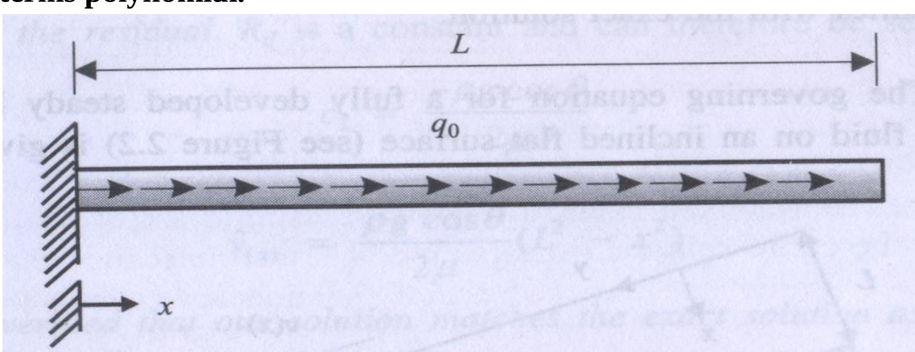
W.K.T

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

substitute a_1 & a_2 value

$$y = \frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi}\right)\sin\frac{\pi x}{l} + \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi}\right)\sin\frac{3\pi x}{l}$$

26. A bar of uniform cross section is clamped at one end and left free at the other end and it is subjected to a uniform axial load p as shown in fig. calculate the displacement and stress in a bar by using two terms polynomial and three terms polynomial. Compare with exact solution.1. displacement of the bar, ∂u. 2. Stress in the bar, σ. By using two term and three terms polynomial.



Solution:

We know that, polynomial function for displacement is,

$U = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots a_nx^n$

Case(i); considering two terms of polynomial,

Apply boundary condtions,

At x=0, u=0

$0 = a_0 + 0$

$a_0 = 0$

substituting a_0 value in equation (1),

$$\frac{du}{dx} = a_1$$

We know that,

Total potential energy of the bar, π=U-H

Where, U=strain energy of the bar,

H=qwork done by external force of the bar.

Strain energy, U=$\frac{EA}{2}\int_0^l (\frac{du}{dx})^2 dx$

$$= \frac{AE}{2}\int_0^l (a_1)^2 dx$$

$$= \frac{AEa_1^{\;2}}{2}[x]_0^l$$

$$U = \frac{EAa1^2\,l}{2}$$

Work done by external force, H=$\int_0^1 pdx$=$\int_0^1 \rho uAdx$

$$H = \rho A\int_0^l udx$$

$$= \rho A\int_0^l a_1 x dx$$

$$= \rho Aa_1\left[\frac{x^2}{2}\right]_0^l$$

$$H = \frac{\rho Aa_1}{2} l^2$$

Substitute U & H value in equation

$$\pi = \frac{AEa_1^{\;2} l}{2} - \frac{\rho Aa_1 l^2}{2}$$

For stationary value π, the following condition must be satisfied

$$\frac{\partial \pi}{\partial a1} = 0$$

$$\frac{AE2a_1 l}{2} - \frac{A\rho l^2}{2} = 0$$

$$AEa_1 l - \frac{A\rho l^2}{2} = 0$$

$$a_1 = \frac{\rho l}{2E}$$

Substitute a_1 value in the equation

$$u = a_1 x$$

$$u = \frac{\rho l}{2E} x$$

W.k.t extension of bar, $\delta u = u_1 - u_0 = \frac{\rho l}{2E} l - 0 = \frac{\rho l^2}{2E}$

$$\delta u = \frac{\rho l^2}{2E}$$

Stress in bar σ = Edu/dx=E $\frac{\rho l}{2E}$

$$\sigma = \frac{\rho l}{2}$$

case ii: Three terms of polynomial,

$$u=a_0+a_1x+a_2x^2$$

apply boundary condition, at x = 0; u = 0

$$0 = a_0 + 0 + 0$$

$$a_0= 0$$

substitute a_0 value in the equation

$$u=a_1x+a_2x^2$$

$$du/dx = a_1x+2a_2x$$

W.K.T,

Total potential energy of the bar, $\pi = U - H$

$$\frac{AE}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{AE}{2}\int_0^l (a_1 + 2a_2 x)^2 dx$$

Strain energy, U = $= \frac{AE}{2}\int_0^l \left(a_1^{\ 2} + 4a_2^{\ 2}x^2 + 4a_1a_2x\right)dx$

$$= \frac{AE}{2}\left(a_1^{\ 2}x + 4a_2^{\ 2}\frac{x^3}{3} + 4a_1a_2\frac{x^2}{2}\right)_0^l$$

$$= \frac{AE}{2}\left(a_1^{\ 2}l + 4a_2^{\ 2}\frac{l^3}{3} + 4a_1a_2\frac{l^2}{2}\right)$$

Work done by external force,

$$H = \int_0^l P dx$$

$$= \int_0^l \rho A u dx$$

$$= \rho A \int_0^l u dx$$

$$= \rho A \int_0^l (a_1 x + a_2 x^2) dx$$

$$= \rho A \left(a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right)_0^l$$

$$= \rho A \left(a_1 \frac{l^2}{2} + a_2 \frac{l^3}{3} \right)$$

$$H = \rho A \left(a_1 \frac{l^2}{2} + a_2 \frac{l^3}{3} \right)$$

Substitute U & H value in equation

$$\pi = \frac{AE}{2} \left(a_1^2 l + 4a_2^2 \frac{l^3}{3} + 2a_1 a_2 l^2 \right) - \rho A \left(a_1 \frac{l^2}{2} + a_2 \frac{l^3}{3} \right)$$

For the stationary value of π, the following conditions must be satitsfied.

$$\frac{\partial \pi}{\partial a_1} = 0 \quad \frac{\partial \pi}{\partial a_2} = 0$$

$$\frac{\partial \pi}{\partial a_1} = \frac{AE}{2} \left[2a_1 l + 0 + 2a_2 l^2 \right] - \rho A \left[\frac{l^2}{2} \right] = 0$$

$$\frac{AE}{2} \left[2a_1 l + 2a_2 l^2 \right] - \rho A \left[\frac{l^2}{2} \right] = 0$$

$$AE \left[a_1 l + a_2 l^2 \right] - \rho A \left[\frac{l^2}{2} \right] = 0$$

$$AE \left[a_1 l + a_2 l^2 \right] = \rho A \left[\frac{l^2}{2} \right]$$

$$\left[a_1 + a_2 l \right] = \left[\frac{\rho l}{2E} \right]$$

$$\frac{\partial \pi}{\partial a_2} = \frac{AE}{2} \left[0 + \frac{8a_2}{3} l^3 + 2a_1 l^2 \right] - \rho A \left[0 + \frac{l^3}{3} \right] = 0$$

$$\frac{AE}{2} \left[\frac{8a_2}{3} l^3 + 2a_1 l^2 \right] = \rho A \left[\frac{l^3}{3} \right]$$

$$\left[\frac{8a_2}{3}l^3 + 2a_1l^2\right] = 2\rho\left[\frac{l^3}{3E}\right]$$

$$\left[\frac{4a_2}{3}l^3 + a_1l^2\right] = \rho\left[\frac{l^3}{3E}\right]$$

$$\left[\frac{4a_2}{3}l + a_1\right] = \left[\frac{\rho l}{3E}\right]$$

$$a_1 + 1.33a_2 l = \frac{\rho l}{3E}$$

Solving the equations

a1 = $\frac{\rho l}{E}$

$a_2 = -\frac{\rho}{2E}$

W.k.t, $u=a_1x+a_2x^2$

Substitute a_1 & a_2 value in the above eqn

$$u = \frac{\rho}{E}\left[lx - \frac{x^2}{2}\right]$$

at x = l, u = u_1 substitute in the above eqn

$$u_1 = \frac{\rho}{E}\left[l^2 - \frac{l^2}{2}\right]$$

$$= \frac{\rho}{E}\left[\frac{l^2}{2}\right]$$

$$= \left[\frac{\rho l^2}{2E}\right]$$

W.k.t extension of bar, $\delta u = u_1 - u_0 = \frac{\rho l^2}{2E} - 0 = \frac{\rho l^2}{2E}$

$$\delta u = \frac{\rho l^2}{2E}$$

for the equation $u = \frac{\rho}{E}\left[lx - \frac{x^2}{2}\right]$

we know that $u = \frac{\rho}{E}\left[lx - \frac{x^2}{2}\right]$

du/dx = $\frac{\rho}{E}[l - x]$

$$\text{Stress in bar } \sigma = Edu/dx = E\,\frac{\rho(l - x)}{E}$$

$$\sigma = \rho(l\text{-}x)$$

27. Consider the differential equation for a problem such as $\frac{d^2y}{dx^2}$+300x^2=0; , 0$\le$x$\le$1 with the boundary condition, y(0)=y(1)=0, the functional corresponding to this problem to be extremized is given by I= $\int_0^1\{-\frac{1}{2}\,(\frac{dy}{dx})^2+300\,x^2+y\}\,dx$. Find the solution of the problem using Rayleigh ritz method using a one term solution is y=ax (1-x^3). (Nov/Dec 2009)

Given: differential equation.

$$\frac{d^2y}{dx^2}+300x^2=0;\ 0<=x<=1$$

Boundary conditions; y(0)=y(1)=0

(i) x=0, y=0

(ii) x=1, y=0

$$I=\int_0^1\{-\frac{1}{2}\,(\frac{dy}{dx})^2+300\,x^2+y\}\,dx$$

Trial function, y=a x (1-x^3)

To find: solution of the problem by using rayliegh- ritz method.

Solution: Trial function, y = a x (1 - x^3)

$$y = ax - ax^4$$

It satisfies the boundary conditions,

x=0, y=0,

x=1, y=0

Differential equation, $\frac{d^2y}{dx^2}+300\,x^2=0$

$$dy/dx=a-4ax^3$$

$$(dy/dx)^2 = (a - 4ax^3)^2$$

We know that,

$$I=\int_0^1\{-\frac{1}{2}\,(\frac{dy}{dx})^2+300\,x^2+y\}\,dx$$

Substitute the equation,

$$I=\int_0^1\{-\frac{1}{2}\,(a-4\,ax^3)^2+300\,x^2\,(ax-\ ax^4)\}dx$$

$$= \int_0^1 \left[-\frac{1}{2}\left(a^2 + 16a^2x^6 - 8a^2x^3\right) + \left(300ax^3 - 300ax^6\right) \right] dx$$

$$= -\frac{1}{2}\left[a^2x + 16a^2\frac{x^7}{7} - 8a^2\frac{x^4}{4} \right]_0^1 + \left[300\frac{ax^4}{4} - 300\frac{ax^7}{7} \right]_0^1$$

$$= -\frac{1}{2}\left[a^2 + \frac{16}{7}a^2 - 2a^2 \right] + \frac{300}{4}a - \frac{300}{7}a$$

$$I = -\frac{a^2}{2} - \frac{8a^2}{7} + a^2 + \frac{300a}{4} - \frac{300a}{7}$$

Apply, $\frac{\partial I}{\partial a}$=0

$$\frac{-2a}{2} - \frac{8}{7}(2a) + 2a + \frac{300}{4} - \frac{300}{7} = 0$$

$$-a - \frac{16}{7}a + 2a + \frac{300}{4} - \frac{300}{7} = 0$$

$$\frac{-16a + 7a}{7} = \frac{-2100 + 1200}{28}$$

$$-9a = -900 * \frac{7}{28}$$

$$a = 25$$

A=25

Hence solution is , y=25x (1-x^3)

28. List and Briefly describe the general steps of the finite element method.

<u>BASIC CONCEPT OF THE FINITE ELEMENT ANALYSIS:</u>

General Methods of the Finite Element Analysis

1. Force Method – Internal forces are considered as the unknowns of the problem.

2. Displacement or stiffness method – Displacements of the nodes are considered as the unknowns of the problem.

General Steps of the Finite Element Analysis

Discretization of structure > Numbering of Nodes and Elements > Selection of Displacement function or interpolation function > Define the material behavior by using Strain – Displacement and Stress – Strain relationships > Derivation of element stiffness matrix and equations > Assemble the element equations to obtain the global or total equations > Applying boundary conditions > Solution for the unknown displacements > computation of the element strains and stresses from the nodal displacements > Interpret the results (post processing).

Boundary Conditions

It can be either on displacements or on stresses. The boundary conditions on displacements to prevail at certain points on the boundary of the body, whereas the boundary conditions on stresses require that the stresses induced must be in equilibrium with the external forces applied at certain points on the boundary of the body.

Consideration During Discretization process

Types of element > Size of element > Location of node > Number of elements.

Basic Concepts of Finite Element Method:

- The basic concept behind the Finite element method is “going from part to whole”
- Name **“FINITE ELEMENT”** coined by Clough

Fitting of a number of piecewise continuous polynomials to approximate the variation of the field variable over the entire domain

In the FEM of analysis , a complex region defining a continuum is discretized into simple geometric shapes called finite elements.

The finite element is a small part of the structure.

In 2 Dimensions, it is usually a triangular or a quadrilateral.

In 3Dimensions---a tetrahedron

Simple functions, such as polynomials are chosen in terms of unknown displacement at the nodes to approximate the variation of the actual displacement over each finite element.

The external loading is also transformed into equivalent force applied at the nodes, later each elements are assembled, then unknowns are calculated by applying boundary conditions.

Flowchart of finite element Analysis (FEA)

Discretization into finite element

↓

Selection and Numbering of nodes and elements

↓

Selection of nodal displacement function in terms of nodal displacement

↓

Forming element stiffness matrix

↓

By applying boundary conditions solving the simultaneous equations

↓

Interpretation of the results

Steps involved in FEA

Step 1: Discretization of Continuum

Step 2: Generation of basic data

- **a)** Numbering of nodes and elements
- **b)** Degree of freedom

Step 3: Determination of element stiffness matrix

Step 4:Assembly of overall stiffness matrix

- **a)** Equilibrium of forces
- **b)** Compatibility of displacement
- **c)** Procedure for assembling overall stiffness matrix

Step 5: Elimination of Restrained Degrees of freedom

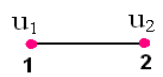
Step 6: Calculation of Nodal Displacement and stresses

- Method of Weighted Residual method- Galerkin method
- Gaussian Elimination method for solving matrices

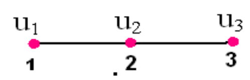
Step 1: Discretization of Continuum

In this step the given structure is divided into subdivisions or elements. Depending upon the problem we may choose I D, II D or IIID elements.

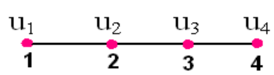
I D elements



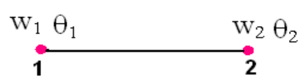
2 NODED LINEAR ELEMENT



3 NODED QUADRATIC ELEMENT

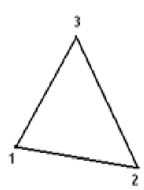


4 NODED CUBIC ELEMENT

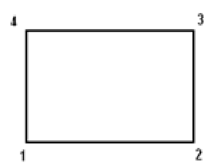


2 NODED BEAM ELEMENT

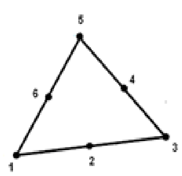
II D elements



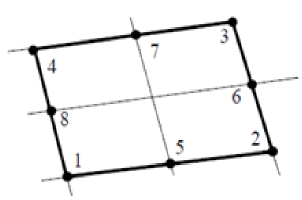
Constant strain triangular element



Bilinear Rectangular element



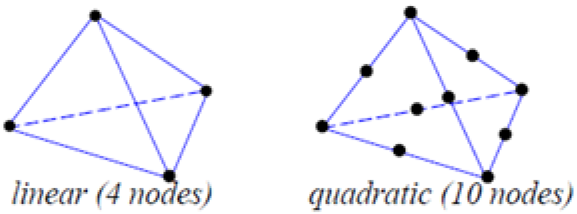
Linear strain triangular element



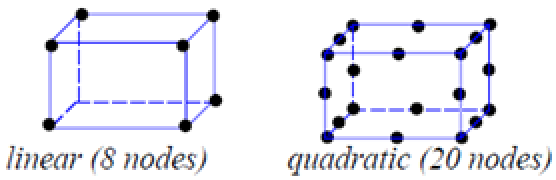
Eight noded quadratic quadrilateral elements

III D elements

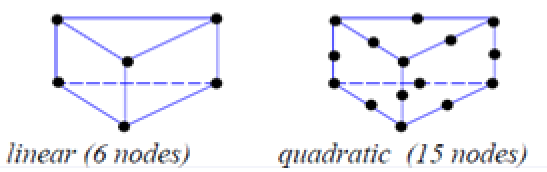
Tetrahedron:



Hexahedron (brick):



Penta:



We make an assumption as to the variation of the unknown solutions over the element. In general, the field variable (example, temperature, displacement etc) is assumed to vary linearly or quadratically or cubically.

Displacement model associated with each element

For $n = 1$ (Linear model)

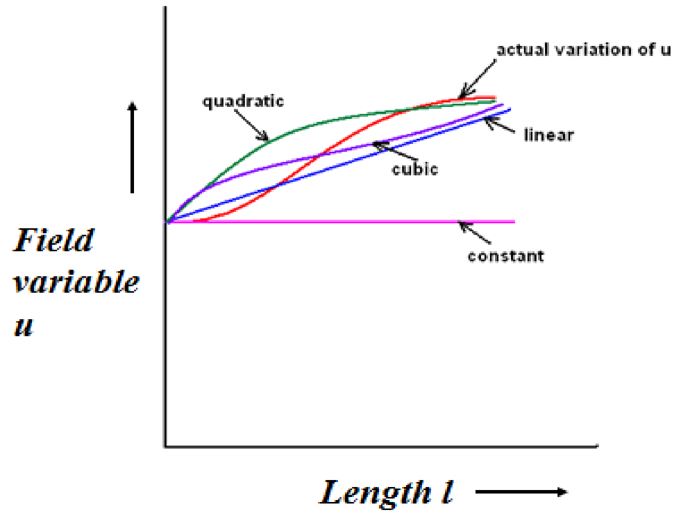
$$u(x) = a_0 + a_1 x$$

For $n = 2$ (quadratic model)

$$u(x) = a_0 + a_1 x + a_2 x^2$$

For $n = 3$ (cubic model)

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$



Derivation of elemental matrices and load vectors:

From the assumed displacement model, the elemental stiffness matrix $[K]^e$ and load vector $[P]^e$ of the element are to be derived using either equilibrium methods or a suitable variational principle.

Assembly of elemental equations to obtain overall stiffness matrix: the individual element stiffness matrices and load vectors are to be assembled in a suitable manner to get the overall stiffness equation which is expressed as

$[K]\{u\} = \{P\}$

where [K] is the assembled stiffness matrix

{u} is the vector of unknowns or nodal displacements

{P} is the vector of nodal forces for the complete structure

Imposition of boundary conditions: The Boundary conditions could now be incorporated to get the reduced equations.

Determination of element stiffness matrix:

Element Stiffness Matrix, k = $\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Solutions for the unknown nodal displacements: The elemental matrices, on assembly, yield a set of equations, which could be expressed as a set of matrices, which could be solved using any iterative procedure or numerical method.

$\{F\} = [K]\ \{\Delta\}$

Computation of elemental strains and stresses: From the unknown displacements, the element strains and stresses can be computed by using the necessary equations of solid or structural mechanics.

30.Write short notes on advantages, disadvantages and applications of Finite Element Method.

Advantages of Finite Element Method

1. FEM can handle irregular geometry in a convenient manner.
2. Handles general load conditions without difficulty
3. Non – homogeneous materials can be handled easily.
4. Higher order elements may be implemented.

Disadvantages of Finite Element Method

1. It requires a digital computer and fairly extensive
2. It requires longer execution time compared with FEM.
3. Output result will vary considerably.

Applications of Finite Element Analysis

Structural Problems:

1. Stress analysis including truss and frame analysis
2. Stress concentration problems typically associated with holes, fillets or other changes in geometry in a body.
3. Buckling Analysis: Example: Connecting rod subjected to axial compression.
4. Vibration Analysis: Example: A beam subjected to different types of loading.

Non - Structural Problems:

1. Heat Transfer analysis:Example: Steady state thermal analysis on composite cylinder.
2. Fluid flow analysis:Example: Fluid flow through pipes.

Two Marks Question and Answers.

UNIT-1INTRODUCTION

1. What is meant by finite element?

A small units having definite shape of geometry and nodes is called finite element.

2. What is meant by finite element analysis? (Nov 2008)

Finite Element method is a numerical method for solving problems of engineering and mathematical physics.

In the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and combine them to obtain the solution of the whole body.

3. State the need of Weak Formulation (Nov 2010)

It reduces the continuity requirement on the trial function assumed in the solution. So it is referred to as the weak form. It is possible to have a wider choice of trial functions

4. What is meant by node or joint?

Each kind of finite element has a specific structural shape and is inter- connected with the adjacent element by nodal point or nodes. At the nodes, degrees of freedom are located. The forces will act only at nodes at any others place in the element.

5. What is the basic of finite element method?

Discretization is the basis of finite element method. The art of subdividing a structure in to convenient number of smaller components is known as Discretization.

6. Give Examples for the finite element.

1. 1-D dimensional Elements: Truss , Bar Element
2. 2-D dimensional Elements: Triangular, Rectangular Element
3. 3-D dimensional Elements: Tetrahedral, Hexahedral Element

7. What are the types of boundary conditions? (Nov 2011)

- Primary boundary conditions
- Secondary boundary conditions

8. State the methods of engineering analysis

- Experimental methods
- Analytical methods
- Numerical methods or approximate methods

9. What are the types of element?

- 1D element
- 2D element
- 3D element

10. State the three phases of finite element method.

- Preprocessing
- Analysis
- Post Processing

11. What is structural problem?

Displacement at each nodal point is obtained. By these displacements solution stress and strain in each element can be calculated.

12. What is non structural problem?

Temperature or fluid pressure at each nodal point is obtained. By using these values properties such as heat flow fluid flow for each element can be calculated.

13.What are the methods are generally associated with the finite element analysis?

- Force method
- Displacement or stiffness method.

14. Explain stiffness method.

Displacement or stiffness method, displacement of the nodes is considered as the unknown of the problem. Among them two approaches, displacement method is desirable.

15. Explain Force Method?

In force Method, internal forces are considered as the unknowns of the problem

16. Why polynomial type of interpolation functions are mostly used in FEM? (May 2013)

The polynomial type of interpolation fuctions are mostly used due to the following reasons:

a. It is easy to formulate and computerize the finite element equations.
b. It is easy to perform differentiation or integration
c. The accuracy of the results can be improved by increasing the order of the polynomial

17. What is meant by post processing?

Analysis and evaluation of the solution result is referred to as post processing. Postprocessor computer program help the user to interpret the result by displaying them in graphical form.

18. Name the variation methods.

- Ritz method.
- Ray-Leigh Ritz method.

19. What is meant by degrees of freedom? (may2011)

When the force or reaction act at nodal point node is subjected to deformation. The deformation includes displacement rotation, and or strains. These are collectively known as degrees of freedom

20. What is meant by discretization?

The art of subdividing a structure in to convenient number of smaller components is known as discretization. These smaller components are then put together. The process of uniting the various elements together is called assemblage.

21. What is meant by Assemblage?

These smaller components are then put together. The process of uniting the various elements together is called assemblage.

22. What is Rayleigh-Ritz method? (May 2012)

It is integral approach method which is useful for solving complex structural problem, encountered in finite element analysis. This method is possible only if a suitable function is available.

23. What is Aspect ratio?

It is defined as the ratio of the largest dimension of the element to the smallest dimension. In many cases, as the aspect ratio increases the in accuracy of the solution increases. The conclusion of many researches is that the aspect ratio should be close to unity as possible.

24. What is truss element? (Nov2012)

The truss elements are the part of a truss structure linked together by point joint which transmits only axial force to the element.

25. What are the h and p versions of finite element method?

It is used to improve the accuracy of the finite element method. In h version, the order of polynomial approximation for all elements is kept constant and the numbers of elements are increased. In p version, the numbers of elements are maintained constant and the order of polynomial approximation of element is increased.

26. Name the weighted residual method (Nov 2011)

- Point collocation method
- Sub domain collocation method
- Lest squares method
- Galerkins method.

27. List the two advantages of post processing.

Required result can be obtained in graphical form. Contour diagrams can be used to understand the solution easily and quickly.

28. During discretization, mention the places where it is necessary to place a node?

- Concentrated load acting point
- Cross-section changing point
- Different material interjections point
- Sudden change in point load

29. What is the difference between static and dynamic analysis?

Static analysis: The solution of the problem does not vary with time is known as static analysis
Example: stress analysis on a beam

Dynamic analysis: The solution of the problem varies with time is known as dynamic analysis
Example: vibration analysis problem.

30. Name any four FEA software's.

- ANSYS
- NASTRAN
- COSMOS

31. Differentiate between global and local axes.

Local axes are established in an element. Since it is in the element level, they change with the change in orientation of the element. The direction differs from element to element.

Global axes are defined for the entire system. They are same in direction for all the elements even though the elements are differently oriented.

32. Distinguish between potential energy function and potential energy functional

If a system has finite number of degree of freedom (q1,q2,and q3), then the potential energy expressed as,

$\pi = f(q1, q2, \text{and } q3)$

It is known as function. If a system has infinite degrees of freedom then the potential energy is expressed as

$$\int f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\right) dx$$

33. What are the types of loading acting on the structure?

- Body force (f)
- Traction force (T)
- Point load (P)

34. Define the body force

A body force is distributed force acting on every elemental volume of the body
Unit: Force per unit volume.
Example: Self weight due to gravity

35. Define traction force

Traction force is defined as distributed force acting on the surface of the body.
Unit: Force per unit area.
Example: Frictional resistance, viscous drag, surface shear

36. What is point load?

Point load is force acting at a particular point which causes displacement.

37. What are the basic steps involved in the finite element modeling

- Discretization of structure.
- Numbering of nodes.

38. Write down the general finite element equation.

$\{F\} = [K]\{u\}$
K-Stiffness matrix in N/mm
U-Nodal displacement in mm

39. What is discretization?
The art of subdividing a structure in to a convenient number of smaller components is known as discretization.

40.list the types of nodes? (May 2012)
- Exterior Nodes
- Interior Nodes

41. What is interpolation functions? (May 2012)
The function used to represent the behavior of the field variable with in an element are called interpolation functions.

42. What should be considered during piecewise trial functions? (May 2011)
- Continuity of the field variable at the junctions
- Continuity of the derivative at the junctions are considered

43.Mention the basic steps of Rayleigh-Ritz method? (May 2011)
The basic steps of Rayleigh-Ritz method are
- Assume a displacement field
- Evaluation of the total potential
- Set up and solve the system of equations

44. What do you mean by constitutive law?

For a finite Element, the stress-strain relations are expressed as follows:

$\{\sigma\}= \{D\}\{e\}$

$\{\sigma\}$= Stress in N/m^2
{D}=Stress-Strain relationship matrix

{e}=Strain (No Unit)

St.Anne's College of Engineering & Technology,
Department of Mechanical Engineering

Subject Name	**:**	**FINITE ELEMENT ANALYSIS**
Subject code	**:**	**ME8692**
Year	**:**	**IIIrd year**
Semester	**:**	**VIth semester**

UNIT II
ONE-DIMENSIONAL PROBLEMS

One Dimensional Second Order Equations – Discretization – Element types- Linear and Higher order Elements – Derivation of Shape functions and Stiffness matrices and force vectors- Assembly of Matrices - Solution of problems from solid mechanics and heat transfer. Longitudinal vibration frequencies and mode shapes. Fourth Order Beam Equation –Transverse deflections and Natural frequencies of beams.

1. **Derive the One dimensional Second order equations.**

ONE DIMENSIONAL SECOND ORDER EQUATIONS

OneDimensionalelement

Bar and beam elements are considered as One Dimensional elements. These elements are often used to model trusses and frame structures.

The total potential energy, stress-strain and strain-displacement relationships are used in developing the finite element method for a one-dimensional problem.

For the one-dimensional problem, the stress (σ), strain (e), displacement (u), and loading (P) depends only on the variable x.

The vector u, e, σ are reduced to

$u = u(x)$,

$e = e(x)$,

$\sigma = \sigma(x)$

the stress-strain and strain-displacement relationship are given by

$\sigma \alpha e$

$\sigma = Ee$

$e = du/dx$

$\sigma = F/A$

e=dl/l = change in length/ Original length

where,

E--- Young's modulus , N/mm2

e--- Strain

σ---Stress, N/mm2

For One dimensional problems, the differential volume, dv can be written as

dv = A dx

Bar,Beamand Truss

Bar is a member which resists only axial loads. A beam can resist axial, lateral and twisting loads. A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements.

Stress,StrainandDisplacement

Stress is denoted in the form of vector by the variable x as σx, Strain is denoted in the form of vector by the variable x as ex, Displacement is denoted in the form of vector by the variable x as ux.

Types ofLoading

(1) Body force (f)

Itis distributedforceactingoneveryelementalvolumeofthebody.

Unit isForce/Unitvolume.

Ex:Selfweightduetogravity.

(2)Traction(T)

Itis distributedforceactingonthesurfaceofthebody.
Unitis Force/Unitarea.Butforonedimensionalproblem,unitisForce/Unitlength.
Ex:Frictionalresistance,viscous dragandSurfaceshear.

(3)Pointload(P)

Itis forceactingataparticularpointwhichcauses displacement.
Unit is N
Ex: Ball bearing loads

FiniteElementModelinghas two processes.

(1)Discretizationofstructure

(2)Numberingofnodes.

2. Write short notes on Discretization.

DISCRETIZATION:

The art of subdividing a structure into convenient number of smaller components is known as discretization.

Nodal force:

The force that acts on the each nodal point is called as nodal force.

Nodal Points:

A finite element has a specific structural shape and it is interconnected with an adjacent element by nodal points (or) Nodes.

Degree of freedom:

Nodal points(or) Nodes are subjected to deformation. This deformations includes displacements, rotations, strains. These are collectively called as degree of freedom.

Discretization can be classified into

1) Natural
2) Artificial (continuum)

Natural Discretization: In any structure analysis, truss is considered as a natural system. the various members of the truss forms the elements. These elements are connected at various joints known as nodes.

The truss consists of 13 elements and 8 nodes. There are four freely moving and 2 extreme constrained nodes. The truss is a natural system as there is no possibility either to increase or decrease the number of elements and nodes.

Artificial Discretization:

A single mass of material as found in forging, concrete dam, deep beam and plate are generally called as continuum.

Unlike the truss element which is physically present in the truss in a continuum, the following three elements exits,

- Triangular element
- Rectangular element
- Quadrilateral element

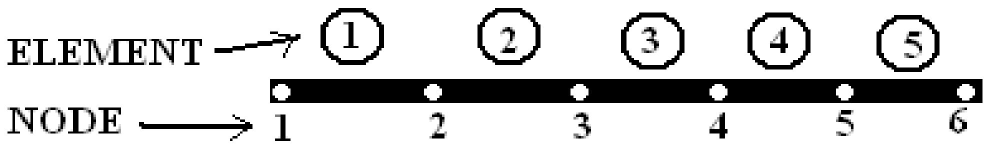
Numbering Scheme:

Each node is allowed to move only in $\pm$x for one dimensional problems. So eah node has one degree of freedom.

A finite element with four nodes has four degree of freedom. Global numbers are those corresponding node numbers on the structure. (i.e) Node numbers are global numbers

The choice of element to be used for discretization depends on the following factors.

1. Number of degree of freedom
2. Shape and size of component
3. Expected accuracy
4. Necessary equations required



3. Discuss about the various element types used in FEA.

<u>ELEMENT TYPE</u>

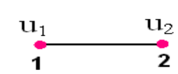
In the FEM of analysis , a complex region defining a continuum is discretized into simple geometric shapes called finite elements.

The finite element is a small part of the structure.

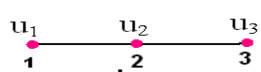
Types of element:

I. One dimensional elements
II. Two dimensional elements
III. Three dimensional elements
IV. Axisymmetric elements

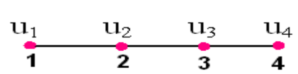
I D elements



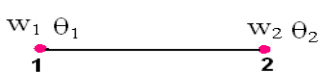
2 NODED LINEAR ELEMENT



3 NODED QUADRATIC ELEMENT

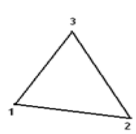


4 NODED CUBIC ELEMENT

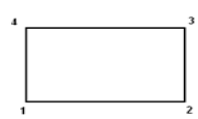


2 NODED BEAM ELEMENT

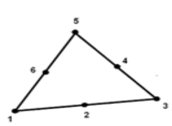
II D elements



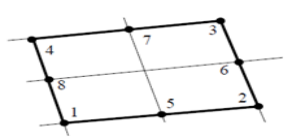
Constant strain triangular element



Bilinear Rectangular element

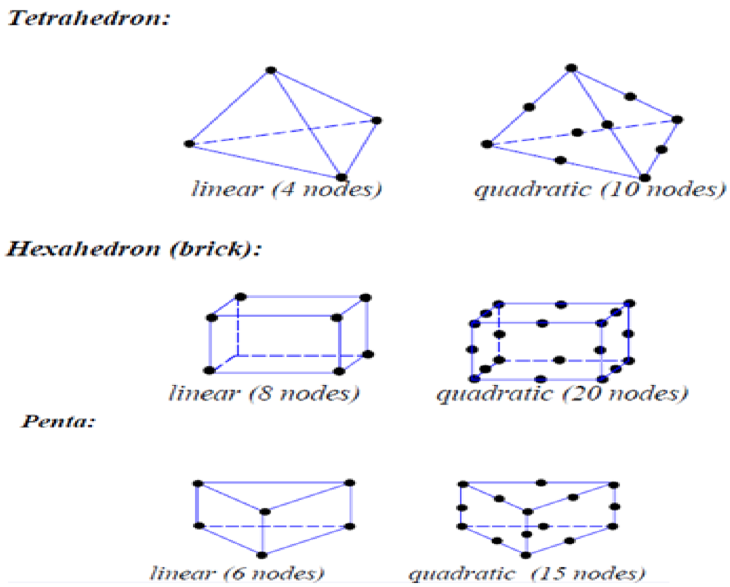


Linear strain triangular element



Eight noded quadratic quadrilateral elements

III D elements



4. **Derive the shape function for three noded triangular elements.**

Shapefunction

The shape function is a function which interpolates the solution between the discrete values obtained at the nodes.

Shape function is used to express the geometry or shape of the element.

Three noded triangular elements:

The field variables are described by the following approximate relation.

$\Phi(x,y) = N_1(x,y)\,\Phi_1 + N_2(x,y)\,\Phi_2 + N_3(x,y)\,\Phi_3$

Φ_1, Φ_2, Φ_3-----field variables

N_1, N_2, N_3 areusuallydenotedasshapefunction.

Shape function has unity value at one nodal point and zero value at other nodal points.

Inonedimensionalproblem,the displacement

$u = \Sigma\, N_i u_i = N_1 u_1$

N_1 – Shape function

u_1 - displacement

For twonoded bar element, the displacement at anypoint within the element is givenby,

$u = \Sigma\, N_i u_i = N_1 u_1 + N_2 u_2$

This can be written as

$u = [N1 \quad N2] \begin{bmatrix} u1 \\ u2 \end{bmatrix}$

For threenodedtriangular element, the displacement at anypoint within the element is givenby,

$$u=\Sigma N_i u_i=N_1u_1+N_2u_2+N_3u_3$$

$$v=\Sigma N_i v_i=N_1v_1+N_2v_2+N_3v_3$$

Shape function need to satisfy the following

(a) First derivatives should be finite within an element;

(b) Displacement should be continuous across the element boundary.

5. Discuss the Characteristics of shape function.

Characteristics of shape function:

- Sum of shape function is equal to one
- First derivative should be finite within the element
- Displacement should be continuous across the element boundary.
- The shape function has unit value at its own nodal point and zero value at other nodal points.
- The shape function for two dimensional elements is zero along each side that the nodes do not touch.
- The shape functions are always polynomials of the same type as the original interpolation equation.
- The accuracy of the result can be improved by increasing the order of the polynomial.

6.Write short notes on stiffness matrix and its characteristics .

Stiffness Matrix [K]

$$\text{Stiffness Matrix [K]} = \int_V [B]^T [D][B] dv$$

Where, dv = A dx

[B] Strain- displacement relationship matrix

[D] Stress strain relationship matrix

In one dimensional problem

Strain e=du/dx

Where, u----Displacement function

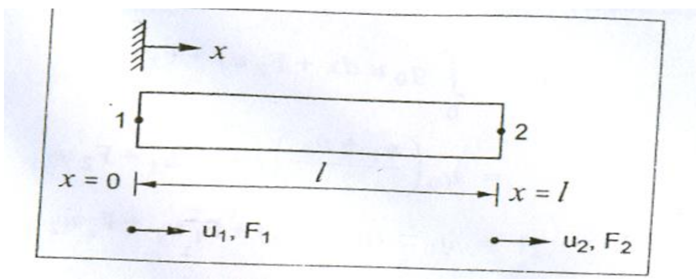
[D] = [E] = E= Young’s Modulus

Character of stiffness matrix:

1. Stiffness matrix is a symmetric matrix.
2. In any column, sum of elements is equal to zero.
3. The determinant is always equal to zero, because, it is an unstable element.
4. N*N is the dimension of the global stiffness matrix.
   Where N---Number of nodes
5. The diagonal coefficients are always positive and relatively large when compared to the off-diagonal values in the same row.

6. **Derive the equation for the bar element formulated from the stationary of a functional.(May/June 2014)**

   Consider a bar element with nodes 1 and 2 as shown in fig. u_1 and u_2 are the displacement at the respective nodes. So u_1 and u_2 are considered as degree of freedom of this bar element. (degree of freedom is nothing but noded displacement)



$$u=N_1u_1+N_2u_2 \text{- - -- } 1$$

$$\text{Where, } N_1=1-\frac{x}{l}$$

$$N_2=\frac{x}{l}$$

Substitute the N_1, N_2 values in equation 1

$$u=(1-\frac{x}{l})u_1+\frac{x}{l}u_2$$

The strain energy stored within the element is given by,

$$u=\int_0^1 \frac{A\,E}{2}\left(\frac{du}{dx}\right)^2 dx$$

$$u=\frac{AE}{2}\left(\frac{u_2-u_1}{l}\right)^2\int_0^l dx$$

$$u = \frac{AE}{2}\left(\frac{u_2 - u_1}{l}\right)^2 (x)_0^l$$

$$u = \frac{AE}{2}\left(\frac{u_2 - u_1}{l}\right)^2 (l)$$

$$u= \frac{AE}{2}\left(\frac{(u_2-u_1)}{l}\right)^2 = (l)$$

When there is distributed force q_0 acting at each point on the element and concentrated forces F at the nodes, the potential of the external forces is given by

$$H = \int_0^l q_0 u dx + F_1 u_1 + F_2 u_2$$

$$= q_0\left(\frac{u_1 + u_2}{2}\right)l + F_1 u_1 + F_2 u_2$$

$$H = q_0 \frac{l}{2}(u_1 + u_2) + F_1 u_1 + F_2 u_2$$

Thus the potential energy

$$\pi = U - H$$

$$\pi = \frac{AE}{2}\frac{(u_2 - u_1)^2}{l} - q_0 \frac{l}{2}(u_1 + u_2) - F_1 u_1 - F_2 u_2$$

Apply, $\frac{\partial \pi}{\partial u_1} = 0$

$$-\frac{AE}{2l} * 2(u_2 - u_1) - \frac{q_0 l}{2} - F_1 = 0$$

$$\frac{AE}{l} * (u_1 - u_2) - \frac{q_0 l}{2} - F_1 = 0$$

$$\frac{AE}{l} * (u_1 - u_2) = \frac{q_0 l}{2} + F_1$$

Similarly , $\frac{\partial \pi}{\partial u_2} = 0$

$$\frac{AE}{2l}*2(u_2-u_1)-\frac{q_0 l}{2}-F_2=0$$
$$\frac{AE}{l}*(u_2-u_1)-\frac{q_0 l}{2}-F_2=0$$
$$\frac{AE}{l}*(u_2-u_1)=\frac{q_0 l}{2}+F_2$$

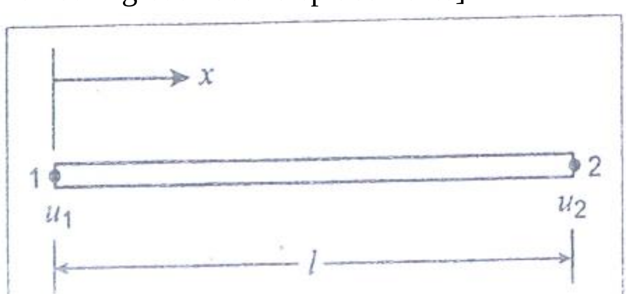
$$\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}\frac{q_0 l}{2}\\ \frac{q_0 l}{2}\end{Bmatrix}+\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

[K] {u}={F}

LINEAR ELEMENT

7. **Derive the displacement function and stiffness matrix for one dimensional linear bar element based on global co-ordinate approach.**

Consider a bar element with nodes 1 and 2 as shown in gig. U1 and u2 are the displacements at the respective nodes. So u_1 and $u_2$are considered as degree of freedom of this bar element. [note: degree of freedom is nothing but nodal displacements]



Since the element has not two degree of freedom. It will two generalized co-ordinates.

$u=a_0+a_1x$ ……… (1)

Where, a_0 and a_1 are global or generalized co0ordinates. Writing the equation in matrix from.

u=[1-x] $\begin{Bmatrix}a_0\\ a_1\end{Bmatrix}$ ………… (2)

At node 1, $u= u_1$, x=0

At node 2, $u= u_2$, x=l

Substituting the above values in equation

$u_1=a_0$ ………………(3)

$u_2= a_0+ a_1 l$ ……………….(4)

Arranging the above equation in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

u* C A

Where, u*- degree of freedom

C-connectivity matrix

A-generalized or global co-ordinates matrix

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1}$$

$$= \frac{1}{l-0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

[Note: $\begin{bmatrix} a_{11} & a_{12}0 \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \dfrac{1}{(a_{11}a_{22} - a_{12}a_{21})} \text{x} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$]

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} 1 & 0 \\ -1 & l \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in equation (2)

$$u= [1\text{-x}] \frac{1}{l} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \ldots\ldots\ldots\ldots(6)$$

$$= \frac{1}{l} \; [1\text{-x}] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} \; [\text{l-x } 0\text{+x}] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

[“matrix multiplication (1x2) x (2x2)=(1x2)]

$$u=[N_1 \; N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Displacement function, $u=N_1 u_1+N_2 u_2$ .... (7)

Where, shape function, $N_1=\dfrac{l-x}{l}$, shape function, $N_2=\dfrac{x}{l}$

We may note that N_1 and N_2 obey the definition of shape function, the function will have a value equal to unity at the node to which it belongs and value at other nodes.

Checking: at node 1, x=0.

$$N_1=\frac{l-x}{l}=\frac{l-0}{l}$$

$$\boxed{N_1=1}$$

$$N_2 = \frac{x}{l} = \frac{0}{l}$$

$$\boxed{N_2 = 0}$$

At node 2, x=l

$$N_1 = \frac{l-x}{l} = \frac{l-l}{l}$$

$$\boxed{N_1 = 0}$$

$$N_2 = \frac{x}{l} = \frac{l}{l}$$

$$\boxed{N_2 = 1}$$

Stiffness matrix

W.K.T, stiffness matrix [k] = $\int_v [B]^T [D][B] dv$

In 1'D bar element,

Displacement function, u = $N_1u_1 + N_2u_2$

Where, $N_1 = \frac{1-x}{l}$; $N_2 = \frac{x}{l}$

W.K.T

Strain displacement,

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{bmatrix}$$

In one dimensional problems, [D] = [E] = E = Young's modulus

Substitute $[B]^T [D][B]$ value stiffness matrix

$$[k] = \int_0^1 \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} * E * \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$[k] = \int_0^1 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} EAdx \qquad \because \text{dv = Adx}$$

$$[k] = EA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^1 dx$$

$$[k] = EA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$[k] = EA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [l]$$

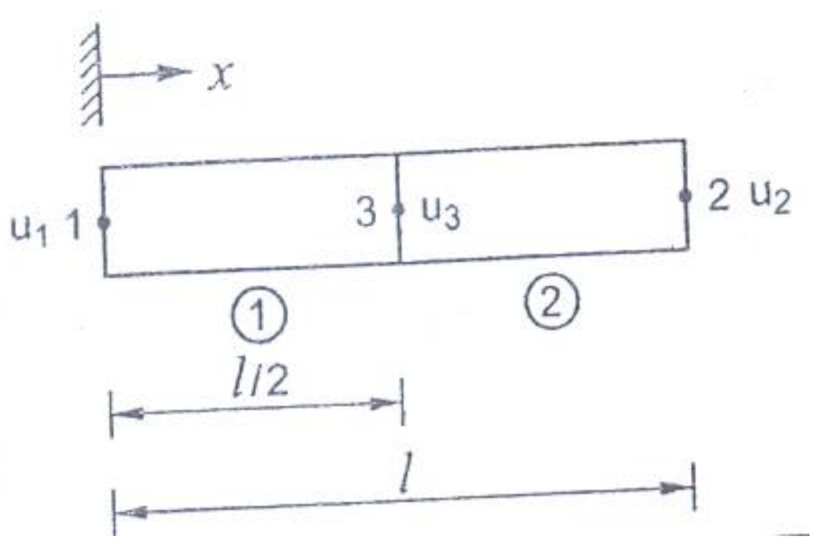
$$\boxed{[k] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

HIGHER ORDER ELEMENT

QUADRATIC ELEMENT:

Derivation of shape functions

8. **Derive the shape functions for one dimensional for one-dimensional quadratic element.(Nov/Dec 2013)**



Consider a quadratic bar element with node 1,2 and 3 as shown in fig. u_1, u_2 u_3 are the displacements at the respective nodes. so u_1, u_2 u_3 are considered as degree of freedom of this quadratic bar element.

Since the element has got three nodal displacement, it will have generalized coordinates.

$$u=a_0+a_1x + a_2x^2 \ldots\ldots(1)$$

where, a_0, a_1 and a_2 are global or generalized coordinates. Writing the equation (1) in matrix form,

$$u=[\,1 \;\; x \;\; x^2\,]\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} \ldots\ldots(2)$$

At node 1, $u=u_1$ $x=0$

At node 2, $u=u_2$ $x=l$

At node 3, $u= u_3$ $x=l/2$

Substitute the above value in equation (1),

$$u_1=a_0 \ldots\ldots\ldots..(3)$$

$$u_2=a_0+a_1l+a_2\, l^2 \ldots\ldots..(4)$$

$$u_3=a_0+a_1(l/2)+a_2(l/2)^2 \ldots\ldots(5)$$

substitute the equation (3) in equation (4) and (5)

Equation (4) $u_2=u_1+a_1l+a_2l^2 \ldots\ldots\ldots\ldots\ldots\ldots(6)$

Equation (5) $u_3=u_1+\frac{a_1l}{2}+\frac{a_2l^2}{4}$ ………..(7)

Equation (6) $u_2-u_1=\frac{a_1l}{2}+\frac{a_2l^2}{4}$ ………..(8)

Arranging the equation (7) and (8) in matrix form.

$$\begin{Bmatrix} u_2-u_1 \\ u_3-u_1 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix}^{-1} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix}$$

$$= \frac{1}{\left(\frac{l^3}{4} \quad \frac{l^3}{2}\right)} \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix}^{-1} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix}$$

[note$\therefore \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11} \quad a_{22} - \quad a_{12} \qquad a_{13})}$ **X** $\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ **]**

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{\frac{-l^3}{4}} \begin{bmatrix} \frac{l^2}{4} & -l^2 \\ \frac{-l}{4} & l \end{bmatrix} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix} \text{.......(9)}$$

$$a_1 = \frac{-4}{l^3}\left[\frac{l^2}{4}(u_2 - u_1) - l_2(u_3 - u_2)\right] \text{.....(10)}$$

$$a_2 = \frac{-4}{l^3}\left[\frac{-l}{2}(u_2 - u_1) - l_2(u_3 - u_2)\right] \text{.....(11)}$$

Equation

$$a_1 = \frac{-4}{l^3}\left[\frac{l^2 u_2}{4} - \frac{l^2 u_1}{4} - l_2(u_3 - u_1)\right]$$

$$= \frac{4l^2 u_2}{4l^3} + \frac{4l^2 u_1}{4l^3} + \frac{4l^2 u_3}{4l^3} - \frac{4l^2 u_1}{4l^3}$$

$$= \frac{-u_2}{l^3} + \frac{u_1}{l} + \frac{4u_3}{l} - \frac{4u_1}{l}$$

$$a_1 = \frac{-3u_1}{l} - \frac{u_2}{l} + \frac{4u_3}{l} \text{ ....(12)}$$

Equation (11)

$$a_2 = \frac{-4}{l^3}\left[\frac{l^2 u_2}{4} - \frac{l^2 u_1}{4} - l_2(u_3 - u_1)\right]$$

$$= \frac{4l^2 u_2}{4l^3} + \frac{4l^2 u_1}{4l^3} + \frac{4l^2 u_3}{4l^3} - \frac{4l^2 u_1}{4l^3}$$

$$= \frac{-u_2}{l^3} + \frac{u_1}{l} + \frac{4u_3}{l} - \frac{4u_1}{l}$$

$$a_2 = \frac{2}{l^2} u_1 - \frac{2u_2}{l^2} + \frac{4}{l^2} u_3 \text{...........(13)}$$

Arranging the equation in matrix

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \text{………..(14)}$$

Substitute the equation (14) in equation (9)

$$\{u\}=[\,1\ x\ x^2] \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \text{……..(15)}$$

$$\{u\}=[(1-\frac{3}{l}x+\frac{2x^2}{l^2})(\frac{-x}{l}+\frac{2x^2}{l^2})(\frac{4x}{l}-\frac{4x^2}{l^2})] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\}=[N_1\ N_2\ N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$\{u\}= N_1 u_1 = N_2 u_2 + N_3 u_3$

Where, shape functions,

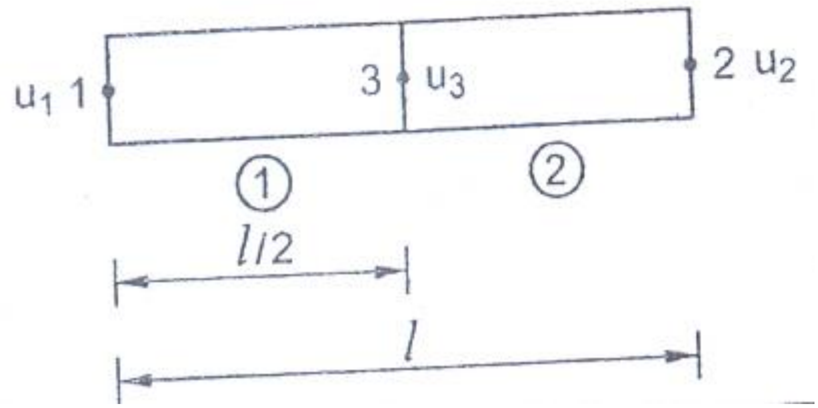
$N_1=1-\frac{3x}{l}+\frac{2x^2}{l^2}$

$N_2=\frac{-x}{l}+\frac{2x^2}{l^2}$

$N_3=\frac{4x}{l}-\frac{4x^2}{l^2}$

Derivation of stiffness matrices:

9. **Derive the stiffness matrix for one-dimensional quadratic bar element.(Nov/Dec 2013)**



Consider a one dimensional quadric bar element with nodes 1, 2 and 3 as shown in fig. let u_1, u_2 and $u_3$be the nodal displacement parameters or otherwise known degree of freedom.

We know that, stiffness matrix, [k]= $\int v[B]^T[D][B]dv$

In one dimensional quadratic bar element,

Displacement function, $u=N_1u_1=N_2u_2+N_3u_3$

Where, $N_1=1-\frac{3x}{l}+\frac{2x^2}{l^2}$

$N_2=\frac{-x}{l}+\frac{2x^2}{l^2}$

$N_2=\frac{4x}{l}-\frac{4x^2}{l^2}$

We know that,

Strain –displacement matrix, [B]= $\frac{dN_1}{dx}\ \frac{dN_2}{dx}\ \frac{dN_3}{dx}$

$$\frac{dN_1}{dx}=\frac{-3}{l}+\frac{4x}{l^2}$$

$$\frac{dN_2}{dx}=\frac{-1}{l}+\frac{4x}{l_2}$$

$$\frac{dN_3}{dx}=\frac{4}{l}-\frac{8x}{l^2}$$

Substitute the equation.

$$[B]=[\left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{-1}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)]$$

$$[B]^T=\begin{Bmatrix}\left(\frac{-3}{l}+\frac{4x}{l^2}\right)\\ \left(\frac{-1}{l}+\frac{4x}{l^2}\right)\\ \left(\frac{4}{l}-\frac{8x}{l^2}\right)\end{Bmatrix}$$

In one dimensional problem,

[D]=[E] =E=young's modulus

Substitute [B] $[B]^T$ and [D] values in stiffness matrix equation.

$$[k]=\int_0^1\begin{Bmatrix}\left(\frac{-3}{l}+\frac{4x}{l^2}\right)\\ \left(\frac{-1}{l}+\frac{4x}{l^2}\right)\\ \left(\frac{4}{l}-\frac{8x}{l^2}\right)\end{Bmatrix}\left[\left(\frac{-3}{l}+\frac{4x}{l^2}\right)\right]\left(\frac{-1}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)xE\ dv.$$

$$[k]=\int_0^1\begin{bmatrix}\left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{-3}{l}+\frac{4x}{l^2}\right) & \left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{-1}{l}+\frac{4x}{l^2}\right) & \left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)\\ \left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{-1}{l}+\frac{4x}{l^2}\right) & \left(\frac{-1}{l}+\frac{4x}{l^2}\right)\left(\frac{-1}{l}+\frac{4x}{l^2}\right) & \left(\frac{-1}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)\\ \left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right) & \left(\frac{-1}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right) & \left(\frac{4}{l}-\frac{8x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)\end{bmatrix}dx$$

$$[k]=\int_0^1\begin{bmatrix}\left(\frac{9}{l^2}-\frac{12x}{l^3}-\frac{12x}{l^3}+\frac{16x^2}{l^4}\right) & \left(\frac{3}{l^2}-\frac{12x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right) & \left(-\frac{12}{l^2}-\frac{24x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right)\\ \left(\frac{3}{l^2}-\frac{12x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right) & \left(\frac{1}{l^2}-\frac{4x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right) & \left(\frac{-4}{l^2}+\frac{8x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right)\\ \left(\frac{-12}{l^2}+\frac{24x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right) & \left(\frac{-4}{l^2}+\frac{8x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right) & \left(\frac{16}{l^2}-\frac{32x}{l^3}-\frac{32x}{l^3}+\frac{64x^2}{l^4}\right)\end{bmatrix}dx$$

$$[k]=E\,A\begin{bmatrix}\left(\frac{9}{l^2}-\frac{12x}{l^3}-\frac{12x}{l^3}+\frac{16x^2}{l^4}\right) & \left(\frac{3}{l^2}-\frac{12x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right) & \left(-\frac{12}{l^2}-\frac{24x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right)\\ \left(\frac{3}{l^2}-\frac{12x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right) & \left(\frac{1}{l^2}-\frac{4x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right) & \left(\frac{-4}{l^2}+\frac{8x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right)\\ \left(\frac{-12}{l^2}+\frac{24x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right) & \left(\frac{-4}{l^2}+\frac{8x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right) & \left(\frac{16}{l^2}-\frac{32x}{l^3}-\frac{32x}{l^3}+\frac{64x^2}{l^4}\right)\end{bmatrix}_0^1$$

$$[k] = E\,A \begin{bmatrix} \left(\frac{9}{l}-\frac{6}{l}-\frac{6}{l}+\frac{16}{3l}\right) & \left(\frac{3}{l}-\frac{6}{l}-\frac{2}{l}+\frac{16}{3l}\right) & \left(\frac{-12}{l}+\frac{12}{l}+\frac{8}{l}-\frac{32}{3l}\right) \\ \left(\frac{3}{l}-\frac{6}{l}-\frac{2}{l}+\frac{16}{3l}\right) & \left(\frac{1}{l}-\frac{2}{l}-\frac{2}{l}+\frac{16}{3l}\right) & \left(\frac{-4}{l}+\frac{4}{l}+\frac{8}{l}-\frac{32}{3l}\right) \\ \left(\frac{-12}{l}+\frac{12}{l}+\frac{8}{l}-\frac{32}{3l}\right) & \left(\frac{-4}{l}+\frac{4}{l}+\frac{8}{l}-\frac{32}{3l}\right) & \left(\frac{16}{l}-\frac{16}{l}-\frac{16}{l}+\frac{64}{3l}\right) \end{bmatrix}$$

$$[k]=E\,A \begin{bmatrix} \frac{7}{3l} & \frac{1}{3l} & \frac{-8}{3l} \\ \frac{1}{3l} & \frac{7}{3l} & \frac{-8}{3l} \\ \frac{-8}{3l} & \frac{-8}{3l} & \frac{16}{3l} \end{bmatrix}$$

$$[k]=E\,A \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}$$

Derive the Force vector and assembly matrices.

10

FORCE VECTOR (F)

Consider a vertically hanging Bar of length l, uniform cross-section A, density ρ and young's modulus,E. It has the self weight (W) because the bar is hanging via gravity.

The force vector is given by

{F} = ∫[N]T * W

w.k.tself weight

W=ρA dx

For one dimensional bar element

$$N=\left(\frac{l-x}{l}\ \frac{x}{l}\right)$$

$$NT = \begin{bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{bmatrix}$$

$$\{F\}=\int_0^l \begin{bmatrix} 1-\frac{x}{l} \\ \frac{x}{l} \end{bmatrix} \rho A\, dx$$

$$=\rho A\int_0^l \begin{bmatrix} 1-\frac{x}{l} \\ \frac{x}{l} \end{bmatrix} dx$$

$$=\rho A \begin{bmatrix} 1-\frac{l^2}{2l} \\ \frac{l^2}{2l} \end{bmatrix}$$

$$=\rho A \begin{bmatrix} l-\frac{l}{2} \\ \frac{l}{2} \end{bmatrix}$$

$$=\rho A \begin{bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{bmatrix}$$

Force Vecor, $\{F\}= \frac{\rho Al}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

ASSEMBLY OF MATRICES:

Consider vertical hanging bar

Consider bar has 4 elements and 5 nodes

W.k.t, Finite element equation is written as,

{F} = [K] {u}

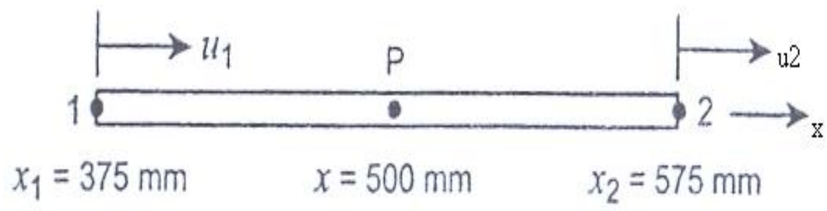
$[K]= \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This is the finite element equation.

11. **Consider a bar as shown in fig. cross sectional area of the bar 750mm^{2}and young's modulus is 2x10^{5}N/mm^2. If u_1=0.5mm and u_2=0.625mm. calculate the following;**
    - **(i) Displacement at point, P**
    - **(ii) Strain, e**
    - **(iii) Stress, σ**

(iv) Element stiffness matrix, [k]. (May/June 2005, April/May 2011)



Given: Area, A=750mm^2

Young's modulus, E=2x10^5 N/mm^2

Displacements, u_1=0.5mm

u_2=0.625mm

Distance, x_1=375mm

X_2=575mm

To find:

(i) Displacement at point, P
(ii) Strain, e
(iii) Stress, σ
(iv) Element stiffness matrix,[k].

Solution:

We know that, actual length of the bar,

$$l=x_2-x_1=575-375$$

$$\boxed{l=200\text{mm}}$$

The distance between point 1 and point p is,

$$X=500-375$$

$$\boxed{x=125\text{mm}}$$

We know that, displacement function for two noded bar element is,

$$u=N_1u_1+N_2u_2$$

Where, shape function, $N_1=\frac{l-x}{l}$

$$N_2=\frac{x}{l}$$

$$N_1=\frac{200-125}{200}$$

$$N_1=0.375$$

$$N_2=\frac{x}{l}=\frac{125}{200}$$

Substitute N_1, N_2, u_1,u_2 values in displacement equation.

We know that, strain e=[B] {u*}

Where, [B] is a stain – displacement matrix.

{u*}- is a degree freedom.

$$[B]=\begin{bmatrix}\frac{-1}{l} & \frac{1}{l}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-1}{200} & \frac{1}{200}\end{bmatrix}$$

$$\text{Strain, } e=[B]\,\{u^*\}=\begin{bmatrix}\frac{-1}{200} & \frac{1}{200}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$=\begin{bmatrix}\frac{-1}{200} & \frac{1}{200}\end{bmatrix}\begin{Bmatrix}.5\\ 0.625\end{Bmatrix}$$

$$=\left[\frac{-1}{200}X\,0.5+\frac{1}{200}X\,0.625\right]$$

$$\text{Strain } e=6.25\text{x}10^{-4}$$

We know that, stress, $\sigma=E\ e=2\text{x}10^5\text{x}6.25\text{x}10^{-4}$

$$\text{Stress, } \sigma=125\ N/mm^2$$

For one dimensional bar element bar element, stiffness matrix is given by,

$$[k]=\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\frac{750\,X\,2\,X\,10^5}{200}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$[K]=7.5x10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

We know that, $\boxed{\text{strain energy, U=}\frac{1}{2}\{u^*\}^T[k]\{u^*\}}$

$$=\frac{1}{2}[u_1\ u_2]\ x\ 7.5x10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$=\frac{1}{2}[0.5\ \ 0.625]x7.5x10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}.5\\ 0.625\end{Bmatrix}$$

$$=\frac{1}{2}[0.5\ \ 0.625]x7.5x10^5\begin{Bmatrix}0.5-0.625\\ -0.5+0.625\end{Bmatrix}$$

$$=\frac{1}{2}\ x7.5x10^5\ [0.5\ \ 0.625]\ \begin{Bmatrix}-0.125\\ 0.125\end{Bmatrix}$$

$$=\frac{1}{2}\ x7.5x10^5\ [0.5x(-0.125)+0.625x0.125]$$

Strain energy, $\boxed{\text{U=5859.37 N-mm}}$

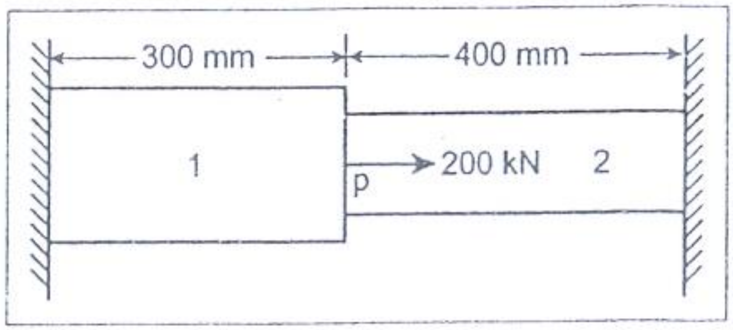
Result:

(i) **U=05781mm**
(ii) **$E=6.25x10^{-4=}$**
(iii) **$\sigma=125N/mm^2$**
(iv) **$[k]=7.5x10^5$**
(v) **U=5859.37N-mm**

12. Consider a bar as shown in fig. an axial load of 200Knis applied at point P. take $A_1=2400mm^2$, $E_1=70x10^9\ N/m^2$. Calculate the following;

(i) **The nodal displacement at point, P.**
(ii) **Stress in each material.**
(iii) **Reaction force.**

(AU May/June 2005, April/May 2011)



Given

Area of element (1), A_1=2400mm^2

Area of lement (2) , A_2=600 mm^{2+}

Length of element (1), l_1=300mm

Length of element (2), l_2=400mm

Young's modulus (1), E_1=70x10^{9}N/m^2

=70x10^3 N/mm^2

Young's modulus (2), E_2=200X10^9 N/m^2

=200x10^3 N/mm^2

Point load, P=200kN=200x10^3 N

To find:

(i) The nodal displacement at point, P.

(ii) Stress in each material.

(iii) Reaction force.

Solution: finite element equation for dimensional two noded bar element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1:

Finite element equation is,

$$\frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{2400X70X10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$1X10^5\begin{bmatrix} a_{11} & a_{12} \\ 5.6 & -5.6 \\ a_{21} & a_{22} \\ -5.6 & 5.6 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

…………. (1)

For element 2: (nodes 2,3) : Finite element equation is,

$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\frac{600X\,200X10^3}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$1X10^5\begin{bmatrix} a_{22} & a_{23} \\ 3 & -3 \\ a_{32} & a_{33} \\ -3 & 3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \quad \ldots\ldots\ldots.. (2)$$

Assemble the finite elemen. Assemble the finite element equations (1) and (2)

$$1X10^5\begin{bmatrix} a11 & a12 & A13 \\ 5.6 & -5.6 & 0 \\ a21 & a22 & a23 \\ -5.6 & 5.6+3 & -3 \\ a31 & a32 & a33 \\ 0 & -3 & 3 \end{bmatrix} 1 \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$1X10^5\underbrace{\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}}_{[k]}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

[note: the bar has 3 nodes. Each node has single degree of freedom. So the global stiffness [k] size is 3x3. The properties of the stiffness matrix are also satisfied.

(i) [k] matrix is symmentric.

(ii) The sum of element in any column is equal to zero.

Applying boundary conditions:

Displacement at nodes 1 and node 3 are zer. So, u_1=u_2=0. A load of 200 acting at node 2.so, F_2 and F_3 values in equation (3)

$$1X10^5\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2X10^5 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1=0$, so, neglect first row and first column of [k] matrix. So, negelectthrird row and third column of [k] matrix. The final reduced equation.

$$1x10^5\ [8.6]\ \{u_2\}=\{2x10^5\}$$

$$8.6x10^5\ u_2=2x10^5$$

$$8.6\ u_2=2$$

$$u_2=0.2325mm$$

stress in each element:

we know that, stress, $\sigma=E\ (du/dx)$

For element (1), stress $\sigma_1=E_1x\ [(u_2-u_1)/l_1]$

$\sigma_1=54.25\ N/mm^2$.

For element (2), stress $\sigma_2=E_2x\ [(u_3-u_2)/l_2]$

$\sigma_2=-116.25\ N/mm^2$.

Reaction force: we know that,

Reaction force, {R}=[k] {u*} – {F}

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1X10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1X10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.2325 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2X10^5 \\ 0 \end{Bmatrix}$$

$$= 1X10^5 \begin{bmatrix} 0-5.6(0.2325+0 \\ 0+8.6(0.2325)+0 \\ 0-3(0.2325)+0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 2X10^5 \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} -1.302X10^5 \\ 0 \\ -0.6975X10^5 \end{bmatrix} - \begin{Bmatrix} 0 \\ 2X10^5 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -1.302X10^5 \\ 0 \\ -0.6975X10^5 \end{Bmatrix}$$

$R_1=-1.302x10^5$

$R_2=0N$

$R_3=-0.6975x10^5$

We know that, reaction is equivalent and opposite to the applied force.

Verification: $R_1+R_2+R_3=-1.302x10^5+0-0.6975x10^5$
$=-200x10^3$ N (applied force)

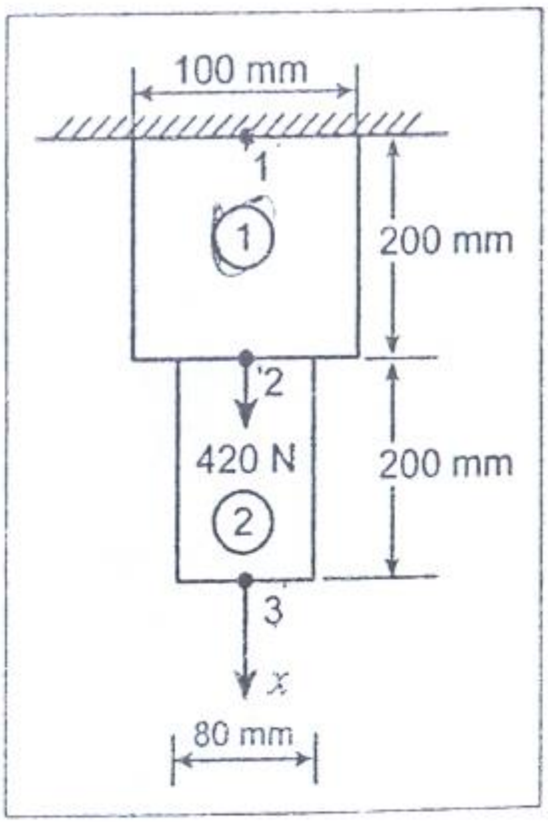
Result:

1. nodal displacement p, $u_2=0.2325$ mm
2. stress in each material, $\sigma_1=54.25$ N/mm^2(tensile)
   $\sigma_2=-116.25$ N/mm^2(compressive)
3. reaction force, $R_1 = -1.302x10^5$; $R_2=0$
   $R3=-0.6975x10^5$ N

13. A thin steel plate of uniform thickness 25mm is subjected to a post load of 420N at mid depth as shown in fig. the plate is also subjected to self-weight. Young's modulus, E=2x10^{5}N/mm^2. And unit weight density, ρ=0.8x10^{-4} N/mm^2. Calculate the following.

(i) Displacement at each nodal point.

(ii) Stresses in each element. (16)



Given:

Thickness t = 25mm

For element 1: Area, $A_1 = 100 * 25 = 2500$ mm^2

For element 2: Area $A_2 = 80 * 25 = 2000$ mm^2

Point load, p = 420 N

Young's modulus, E = 2 * 10^5 N/mm^2

Unit weight density, ρ = 0.8 * 10^{-4} N/mm^3

To find

i. Displacement at each nodal point, u_1, u_2 and u_3
ii. Stress in each element, σ_1 and σ_2

Solution:

The steel plate is subjected self – weight. So, we have to find body force acting at nodal point 1, 2 and 3.

W.k.t, body force vector, $\{F\} = \dfrac{\rho A l}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$

For element 1: force vector $\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \dfrac{\rho_1 A_1 l_1}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$

$$\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \frac{0.8*10^{-4}*2500*200}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = 20\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \begin{Bmatrix}20\\20\end{Bmatrix}$$

For element 2: force vector $\begin{Bmatrix}F_2\\F_3\end{Bmatrix} = \dfrac{\rho_2 A_2 l_2}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$

$$\begin{Bmatrix}F_2\\F_3\end{Bmatrix} = \frac{0.8*10^{-4}*2000*200}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = 16\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\begin{Bmatrix}F_2\\F_3\end{Bmatrix} = \begin{Bmatrix}16\\16\end{Bmatrix}$$

Assembling the force vector,

$$\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix} = \begin{Bmatrix}20\\20+16\\16\end{Bmatrix} = \begin{Bmatrix}20\\36\\16\end{Bmatrix}$$

A point load of 420 N is acting at mid depth, at nodal point 2 as shown in fig, so add 420 N in F_2 vector.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 36+420 \\ 16 \end{Bmatrix}$$

$$\text{Global force vector} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 456 \\ 16 \end{Bmatrix}$$

Finite element equation for one dimensional plate element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: nodes 1, 2

Finite element equation is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A_1 E}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{2500 * 2 * 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 2 * 10^5 \begin{bmatrix} 12.5 & -12.5 \\ -12.5 & 12.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 2: nodes 2, 3

Finite element equation is

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{2000 * 2 * 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 2 * 10^5 \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Assembling the finite element equation 1 and 2

$$2*10^5\begin{bmatrix}12.5 & -12.5 & 0\\ -12.5 & 12.5+10 & -10\\ 0 & -10 & 10\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix}$$

$$2*10^5\begin{bmatrix}12.5 & -12.5 & 0\\ -12.5 & 22.5 & -10\\ 0 & -10 & 10\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix}$$

Apply boundary condition at node 1, displacement $u_1 = 0$, substitute u_1, F_1, F_2 and F_3 values in the above matrix.

$$2*10^5\begin{bmatrix}12.5 & -12.5 & 0\\ -12.5 & 22.5 & -10\\ 0 & -10 & 10\end{bmatrix}\begin{Bmatrix}0\\u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}20\\456\\16\end{Bmatrix}$$

Neglect the first row and first column in the above matrix

$$2*10^5\begin{bmatrix}22.5 & -10\\ -10 & 10\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}456\\16\end{Bmatrix}$$

$$2*10^5(2.25u_2-10u_3)=456$$

$$2*10^5(-10u_2+10u_3)=16$$

$$\boxed{u_3 = 1968 *10^{-4}}$$

$$\boxed{u_2 = 1.88 * 10^{-4}}$$

Stresses in each element

We know that,

Stress, $\boxed{\sigma = E\frac{du}{dx}}$

For element 1: $\sigma_1 = E\dfrac{u_2-u_1}{l_1} = 2*10^5*\dfrac{1.88*10^{-4}-0}{200} = \boxed{0.188 N/mm^2}$

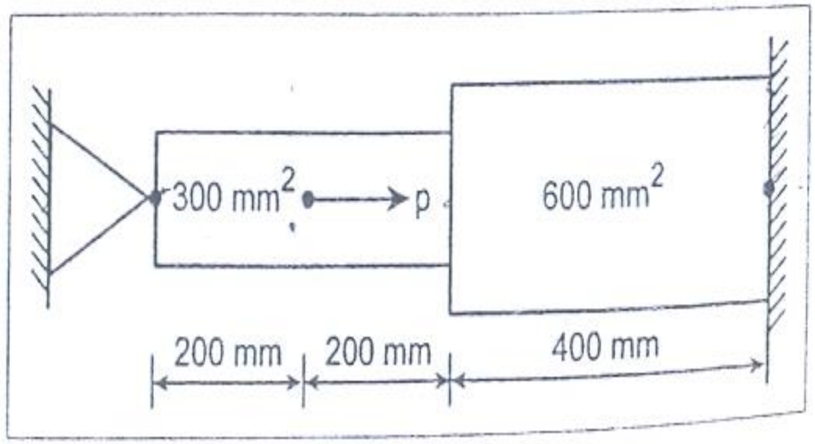
For element 2: $\sigma_2 = E\dfrac{u_3-u_2}{l_2} = 2*10^5*\dfrac{1.968*10^{-4}-1.88*10^{-4}}{200} = \boxed{0.008 N/mm^2}$

14. Consider the bar as shown in fig. Take E = $2*10^5$ N/mm^2; p = 400 kN. Calculate the following.

(i) Nodal displacement

(ii) Element stresses.

(iii) Support reactions.((May/June 2005)



Given:

Area of element 1 (A_1) = 300 mm^2

Area of element 2 (A_2) = 300 mm^2

Area of element 3 (A_3) = 600 mm^2

Length of element 1 l_1 = 200 mm

Length of element 2 l_2 = 200 mm

Length of element 3 l_3 = 400 mm

Young's modulus E = $2*10^5$ N/mm^2

Point load, p = 400 kN

To find

i. Nodal displacements, u_1, u_2, u_3 and u_4
ii. Element stresses, σ_1, σ_2 and σ_3
iii. Reactions at the support R_1 and R_4

Solution:

Finite element equation for one dimensional two noded bar element is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: nodes 1, 2

Finite element equation is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A_1 E}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{300 * 2 * 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 1 * 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 2: nodes 2, 3

Finite element equation is

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{300 * 2 * 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 1 * 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

For element 3: nodes 3, 4

Finite element equation is

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = \frac{A_1 E}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = \frac{600 * 2 * 10^5}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = 1 * 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

Assembling the finite element equation 1, 2 and 3

$$1*10^5\begin{bmatrix}3 & -3 & 0 & 0\\ -3 & 3+3 & -3 & 0\\ 0 & -3 & 3+3 & -3\\ 0 & 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\end{Bmatrix}$$

$$1*10^5\begin{bmatrix}3 & -3 & 0 & 0\\ -3 & 6 & -3 & 0\\ 0 & -3 & 6 & -3\\ 0 & 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\end{Bmatrix}$$

Applying boundary conditions:

i. Node 1 and node 4 are fixed, u_1 and $u_4 = 0$.

j. $400*10^3$ N is acting at node 2. $F_2 = 400*10^3$ N

k. Self weight is neglected, $F_1 = F_3 = F_4 = 0$.

Substituting the boundary values in the above matrix

$$1*10^5\begin{bmatrix}3 & -3 & 0 & 0\\ -3 & 6 & -3 & 0\\ 0 & -3 & 6 & -3\\ 0 & 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}0\\u_2\\u_3\\0\end{Bmatrix}=\begin{Bmatrix}0\\400*10^3\\0\\0\end{Bmatrix}$$

$$1*10^5\begin{bmatrix}6 & -3\\ -3 & 6\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}400*10^3\\0\end{Bmatrix}$$

$$1*10^5(6u_2-3u_3)=400*10^3$$

$$1*10^5(-3u_2+6u_3)=0$$

$$\boxed{u_2 = 0.88mm}$$

$$\boxed{u_3 = 0.44mm}$$

Stresses in each element

We know that,

Stress, $\boxed{\sigma = E\dfrac{du}{dx}}$

For element 1: $\sigma_1 = E\dfrac{u_2-u_1}{l_1} = 2*10^5*\dfrac{0.88-0}{200} = \boxed{888.8N/mm^2}$

For element 2: $\sigma_2 = E\dfrac{u_3-u_2}{l_2} = 2*10^5*\dfrac{0.44-0.88}{200} = \boxed{-444.44N/mm^2}$

For element 3: $\sigma_3 = E\dfrac{u_4 - u_3}{l_3} = 2*10^5 * \dfrac{0-0.44}{400} = \boxed{-222.22 N/mm^2}$

Reaction force in each element

We know that,

Reaction force, {R} = [k] {u*} – {F}

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 1*10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 1*10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.88 \\ 0.44 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 400*10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 1*10^5 \begin{bmatrix} 0-3*0.88+0+0 \\ 0+6*0.88-3*0.44+0 \\ 0-3*0.88+6*0.44+0 \\ 0+0-3*0.44+0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 400*10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 1*10^5 \begin{bmatrix} -2.67 \\ 4 \\ 0 \\ -1.33 \end{bmatrix} - \begin{Bmatrix} 0 \\ 400*10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} -2.67*10^5 \\ 4*10^5 \\ 0 \\ -1.33*10^5 \end{bmatrix} - \begin{Bmatrix} 0 \\ 400*10^3 \\ 0 \\ 0 \end{Bmatrix}$$

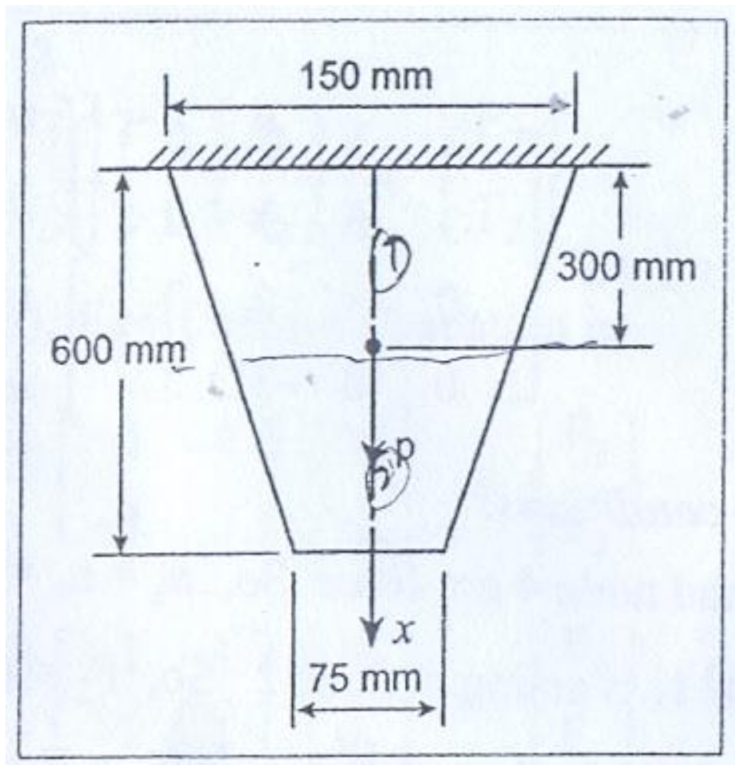
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} -2.67*10^5 \\ 0 \\ 0 \\ -1.33*10^5 \end{bmatrix}$$

$\boxed{R_1 = -2.67 * 10^3 \text{ N}; R_2 = 0; R_3 = 0; R_4 = -1.33*10^5 \text{ N}}$

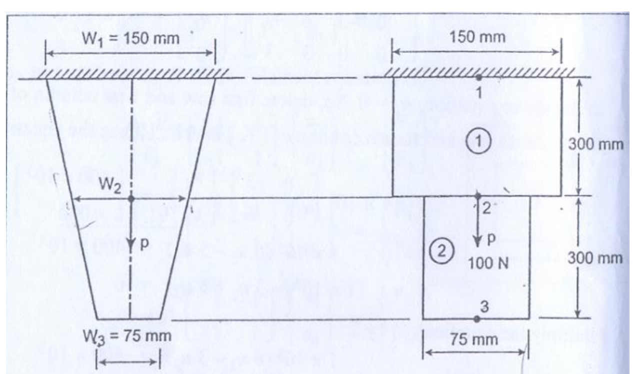
15. Consider a taper steel plate of uniform thickness, t=25mm as shown in fig. the young's modulus of the plate, E=2x10^5 N/mm^2 and weight density, ρ=0.82x10^{-4} N/mm^3. In addition

to its self-weight, plate is subjected to a point load P=100N at its mid point. Calculate the following by modeling the plate with two finite elements.

- **(i) Global force vector { F}**
- **(ii) Global stiffness matrix [k]**
- **(iii) Displacement force at the support.**
- **(iv) Stresses in each element.**
- **(v) Reaction force at the support.(M.E Aero Engg Dec 2006)**



Given: in this problem, the area of the element is varying at each cross-sectional consider this area variation, the problem will be tedious, so. The given taper bar as stepped bar as shown in fig below.



Solution: area at nodal 1, A_1=width x thickness= $W_1 x t_1$=150x75

A_1=3750mm^2

Area at node 2, A_2=W_2xt_2

$$=\left[\frac{W_1+W_3}{2}\right]X\ t_2=\left[\frac{150+75}{2}\right]x2$$

A_2=2812.5mm^2

Area at node 3, A_3 = W_3xt_3=75x25

A_3=1875mm^2

Average area of element (1): $\overline{A}=$ (area at node, 1+area at node, 2)/2

$\overline{A}=$ (3750+2812.5)/2

$\overline{A}_1$ =3281.25mm^2

Average area of element (2): $\overline{A}=$ (area at node, 2+area at node, 3)/2

$\overline{A}=$ (2812.5+1875)/2

$\overline{A}_2$ = 2343.75mm^2

Young's modulus, E=2x10^5 N/mm^2

Weight density, ρ=0.82x10^{-4} N/mm^3, length, l=300mm

Point load, P=100N.

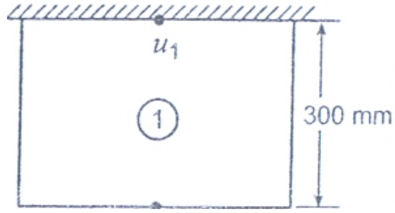
To find:

1. Global force vector { F}
2. Global stiffness matrix [k]
3. Displacement force at the support.
4. Stresses in each element.
5. Reaction force at the support.

Solution: the steel plate is subjected to self-weight. So, we have to find the body force acting at nodal points 1,2 and 3

We know that, body force vector, {F}= $\dfrac{\rho Al}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$

For element (1): force vector, $\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{\rho_1 \overline{A_1} l_1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

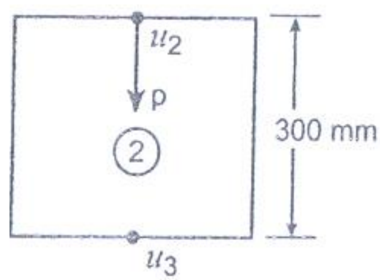


$$= \frac{0.82x10^{-4}\, X\, 3281.25x300}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$=40.359x \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 40.359 \end{Bmatrix}$$

For element (2): force vector, $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{\rho_2 \overline{A_2} l_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$



$$= \frac{0.82x10^{-4}\, X\, 2343.75\, X\, 300}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$=28.828 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 28.828 \\ 28.828 \end{Bmatrix}$$

Assembling the force vector, assemble the equation (1) and (2),

$$\begin{Bmatrix} F_! \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 40.359 + 28.828 \\ 28.828 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 69.187 \\ 28.828 \end{Bmatrix}$$

A point load of 100 N is acting at node 2 as shown in fig. so, add 100N in F_2 vector,

$$\begin{Bmatrix} F_! \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 69.187 + 100 \\ 28.828 \end{Bmatrix}$$

Global force vector, $\begin{Bmatrix} F_! \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$

Finite element equation for one dimensional plate element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For elment1: (node 1,2) finite element equation is,

$$\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{3281.25 X 2 X 10^5}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$10.937 X 2 X 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$2 X 10^5 \begin{bmatrix} 10.937 & -10.937 \\ -10.937 & 10.937 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \ldots\ldots\ldots.(4)$$

For element2 (nodes 2, 3): finite element equation is ,

$$\frac{\overline{A_2} E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\frac{2343.75 X\, 2 X 10^5}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$7.8125 X\, 2 X 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$2 X 10^5 \begin{bmatrix} 7.8125 & -7.8125 \\ -7.8125 & 7.8125 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \ldots\ldots\ldots\ldots(5)$$

Assembling the finite element equation

$$2 X 10^5 \begin{bmatrix} 10.937 & -10.937 & 0 \\ -10.937 & 10.937 + 7.8125 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$2 X 10^5 \begin{bmatrix} 10.937 & -10.937 & 0 \\ -10.937 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Apply the boundary conditions, at node 1, displacement u_1=0 substitute and F_3 values in equation (6)

$$2X10_5 \begin{bmatrix} 18.749 & -7.8125 \\ -7.8125 & 7.8125 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 169.187 \\ 28.828 \end{Bmatrix}$$

$2X10^5(18.749u_2-7.8125u_3)=169.187$

$\underline{2X10_5(-7.8125u_2=7.8125u_3)=28.828}$

Solving, $2x10_5x(10.936)u_2=198.015$

$u_2=9.053x10^{-5}$mm

Substitute u_2 value in equation (7),

$2x10^5\ [18.749(9.653x10^{-5})-7.8125u]=169.187$

$18.749x9.053x10^{-5}-7.8125u_3=-8.514x10^{-4}$

$u_3=10.898x10^{-5}$

We know that, stress, σ=0.060 N/mm^2

$$\sigma_1 = E x \frac{u_2 - u_1}{l_1} = 2x10^5 x \frac{(9.053X10^{-5} - 0)}{300}$$

$$= \frac{2X10^5 X(10.898X10^{-5} X 9.053X10^{-5})}{300}$$

σ_2=0.0123 N/mm^2

Reaction force: we know that,

Reaction force, {R} = [k] {u*}-{F}

$$\begin{Bmatrix} R_! \\ R_2 \\ R_3 \end{Bmatrix} = 2x10^5 \begin{bmatrix} 10.937 & -10.937 & 0 \\ -10.937 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$= 2x10^5 \begin{bmatrix} 10.937 & -10.937 & 0 \\ -10.937 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix} \begin{Bmatrix} 0 \\ 9.53X10^{-5} \\ 10.898X10^{-5} \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$= 2x10^5 \begin{Bmatrix} -9.901X10^{-4} \\ 8.459X10^{-4} \\ 1.441X10^{-4} \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$= \begin{Bmatrix} -198.02 \\ 169.187 \\ 28.828 \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$\begin{Bmatrix} R_! \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -238.739 \\ 0 \\ 0 \end{Bmatrix}$$

R_1=-238.379

R_2=0 N

R_3=0 N

We know that, reaction force is equivalent and opposite to the applied force.

Verification: R1+R_2+R_3= -238.379+0+=-238.379N

Applied force $F_1+F_2F_3$=40359+169.187+28.828=238.37N

Result:

(i) $\{F\}=\begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$

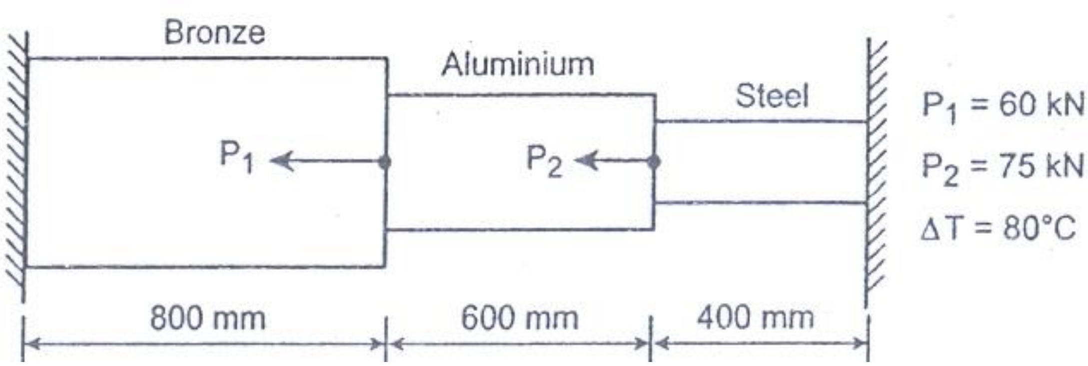
(ii) $[k]=2X10^5\begin{bmatrix} 10937 & -10.937 & 0 \\ -10.937 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix}$

(iii) $u_1=0$
$u_2=9.053\text{x}10^{-5}$ mm
$u_3=10.989\text{x}10^{-5}$mm

(iv) $\sigma_1=0.060$ N/mm^2
$\sigma_2=0.0123$ N/mm^2

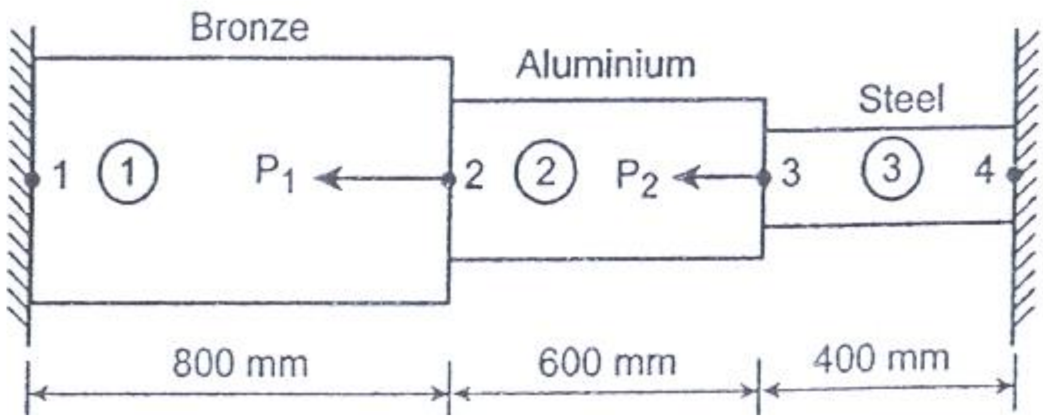
(v) $R_1=-238.739$N
$R_2=0$ N.
$R_3=0$ N.

16. The structure shown in fig. is subjected to an increase in temperature of 80◦C. determine the displacements, stresses and support reactions. Assume the following data; (Jan 2007,M.E(Engg. Design)



Bronze	Aluminium	Steel
A=2400mm^2	1200 mm^2	600 mm^2
E=83Gpa	70Gpa	200Gpa
A=18.9x10^{-6}/◦C	23 x10^{-6}/◦C	11.7 x10^{-6}/◦C

Given:



P_1=60kN=-60X10^3N

P_2=75kN=-75X10^3N

$\Delta T = 80 \circ C$

l_1=800mm

l_2=600mm

l_3=400mm

For bronze: A_1=2400mm^2

E_1=86Gpa=83x10^3 N/mm^2

=83x10^9 N/m^2=83x10^3pa

α_1=18.9x10^{-6}/$\circ C$

For aluminum:

A_2=1200mm^2

E_2=70Gpa=83x10^3 N/mm^2

=70x10^9 N/m^2=70x10^3pa

α_2=23x10^{-6}/$\circ C$

For steel: A_3=2400mm^2

E_3=200Gpa=200x10^3 N/mm^2

=200x10^9 N/m^2=83x10^3pa

α_3=11.7x10^{-6}/$\circ C$

To find:

1. Displacement at each node, u_1, u_2, u_3 and u_4
2. Stresses in each element, σ_1,σ_2,σ_3.
3. Support reaction, R_1 and R_4.

Solution: finite element equation for the one dimensional two noded bar element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: (node 1.2):

Finite element equation is,

$$\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{2400X83X10^3}{800}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$10^3\begin{bmatrix} 249 & -249 \\ -249 & 249 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

For element 2: (node 2, 3)

Finite element equation is,

$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\frac{1200X70X10^3}{600}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$10^3\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

For element 3: (node 3, 4)

Finite element equation is,

$$\frac{A_3E_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}=\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{600X200X10^3}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}=\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$10^3\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}=\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

Assembling the equation (1) (2), and (3)

$$10^3\begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 249+140 & -140 & 0 \\ [0] & -140 & 140+300 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} X \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$10^3\begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 389 & -140 & 0 \\ [0] & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} X \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Assembling the {F} matrix:

We know that, load factor, $\{F\} = EA\alpha\,\Delta\,T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

For element (1): $\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = E_1A_1\ \alpha_1\,\Delta\,_1T_1 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$$= 83X10^3X2400X18.9X10^{-6}X80X \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} -301.1904 \\ 301.1904 \end{Bmatrix}$$

For element (2): $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = E_2A_2\ \alpha_2\,\Delta\,_2T_2 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$$=70 \text{ x } 10^3 \text{ x } 1200 \text{ x } 23 \text{ x}10^{-6} \text{x}80 \text{ x } \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} -154.56 \\ 154.56 \end{Bmatrix}$$

For element (3): $\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = E_3A_3\ \alpha_3\,\Delta\,_2T_3 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$$=200 \text{ x } 10^3 \text{ x } 600 \text{ x } 11.7 \text{ x}10^{-6} \text{x}80 \text{ x } \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} -112.32 \\ 112.32 \end{Bmatrix}$$

Assemble the equation (5), (6) and (7)

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^4 X \begin{Bmatrix} -301.1904 \\ 301.1904 - 154.56 \\ 154.56 - 112.32 \\ 112.32 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^4 \text{ X} \begin{Bmatrix} -301.1904 \\ 146.6304 \\ 42.24 \\ 112.32 \end{Bmatrix}$$

From the fig, we know that, axial load of $60 \text{x} 10^3$= N is acting at node 2 and $75 \text{x} 10^3$N is act in F_2 at node 3, so, subtract $60 \text{x} 10^3$ N in F_3 vector.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^4 \text{ X} \begin{Bmatrix} -301.1904 \\ 146.6304-60 \\ 42.24-75 \\ 112.32 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^4 \text{ X} \begin{Bmatrix} -301.1904 \\ 86.6304 \\ -32.76 \\ 112.32 \end{Bmatrix}$$

Substitute equation (8) in equation (4),

$$10^3 \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 389 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = 10^3 \text{X} \begin{Bmatrix} -301.1904 \\ 86.6304 \\ -32.76 \\ 112.32 \end{Bmatrix}$$

Applying the boundary condition, at node 1, u_1=0 and node 4, u_2=0, substitute $u_1$and u_4 values in the above equation.

$$\begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 3898 & 0 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -301.1904 \\ 86.6304 \\ -32.76 \\ 112.32 \end{Bmatrix}$$

In the above equation, u_1=0. So, neglect first row and first column of [k] matrix. u_4=0, so, neglect fourth row and forth column of [k] matrix. hence the equation reduce to

$$\begin{bmatrix} 389 & -140 \\ -140 & 440 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 86.6304 \\ -32.76 \end{Bmatrix}$$

$389u_2-140u_3=86.6304$

$-140u_2+440u_3=-32.76$

$1222.571u_1-440u_3=272.2707$

$\underline{-140u_2+440u_3=-32.76}$

Solving, $1082.571u_2=239.51$

u_2=0.2212 mm

Substitute u_2 value in equation,

$-140(0.2212)+440u_3=-32.44$

$u_3=-0.00345mm$

we know that, thermal stress, $\sigma=E\frac{du}{dx}-E\alpha\,\Delta T$

for element (1), $\sigma_1=\frac{E_1(u_2-u_1)}{l_1}-E_1\alpha_1\Delta T$

$$=\frac{83X10^3(0.2212-0)}{800}-83X10^3X18.9X10^{-6}X80$$

$\sigma_1=-102.5455\ N/mm^2$

for element 2: $\sigma_2=\frac{E_2(u_3-u_2)}{l_2}-E_2\alpha_2\Delta T$

$\sigma_2=-155.009\ N/mm^2$

for element 3:

$$\sigma3=\frac{E_3(u_4-u_3)}{l_3}-E_3\alpha_3\Delta T$$

$\sigma_3=-185.475\ N/mm^2$.

we know that, reaction force {R}=[k] {u*} –{F}

$$\begin{Bmatrix}R_1\\R_2\\R_3\\R_4\end{Bmatrix}=10^{3=}\begin{bmatrix}249&-249&0&0\\-249&389&-140&0\\0&-140&-300&0\\0&0&-300&300\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}-\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\end{Bmatrix}$$

$$=10^3\begin{bmatrix}249&-249&0&0\\-249&389&-140&0\\0&-140&-300&0\\0&0&-300&300\end{bmatrix}\begin{Bmatrix}0\\0.2212\\-0.00345\\0\end{Bmatrix}-10^3\begin{Bmatrix}-301.1904\\86.6304\\-32.75\\112.32\end{Bmatrix}$$

$$=10^3\begin{bmatrix}0-249X0.2212+0+0\\0+389X0.2212-140X(-0.00345)+0\\0-140X0.2212+440X(-0.00345)+0\\0+0-300X(-0.00345)+0\end{bmatrix}-10^3\begin{Bmatrix}-301.1904\\86.6304\\-32.75\\112.32\end{Bmatrix}$$

$$=10^3\begin{Bmatrix}-55.0788\\86.5018\\-32.486\\1.035\end{Bmatrix}-10^3\begin{Bmatrix}-301.1904\\86.6304\\-32.76\\112.32\end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 10^3 X \begin{Bmatrix} 246.1116 \\ 0 \\ 0 \\ -113.35 \end{Bmatrix}$$

R_1=246.1116X10^3N

R_2=0

R_3=0

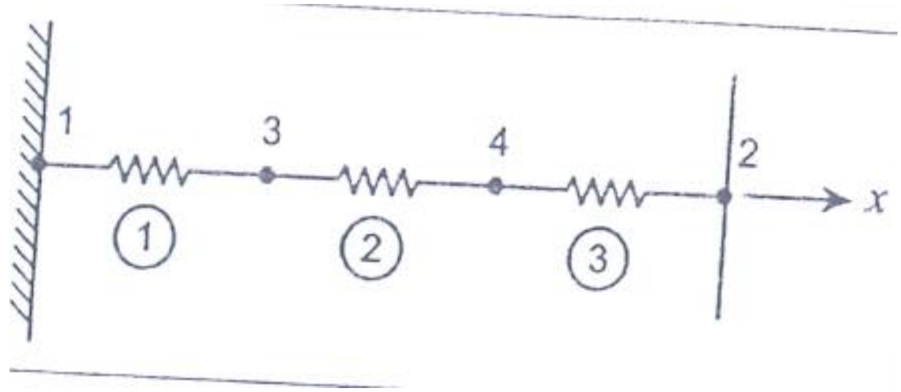
R_4=-113.35X10^3 N.

17. A spring assemblage with arbitrary numbered nodes are shown in fig. the nodes 1 and 2 arefixed and a force of 500kN is applied at node 4 in the x –directions. Calculate the following.

(i) Global stiffness matrix.

(ii) Nodal displacement

(iii) Reactions at each nodal point.



Given:

Nodal force, F_4 = 500kN

Spring constant, k_1 =100 kN/m; k_2 = 200 kN/m; k_3 = 300 kN/m

To find:

i) Global stiffness matrix [k]

ii) Nodal displacements, u_1, u_2, u_3 and u_4

iii) Reactions at each nodal point, R_1, R_2, R_3 and R_4

Solution:

Finite element equation for spring element is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: node 1,3, the finite element equation is

$$k_1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_3\end{Bmatrix}$$
$$100\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_3\end{Bmatrix}$$
$$\begin{bmatrix}100 & -100\\ -100 & 100\end{bmatrix}\begin{Bmatrix}u_1\\ u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_3\end{Bmatrix}$$

For element 2: node 3,4, the finite element equation is

$$k_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$
$$200\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$
$$\begin{bmatrix}200 & -200\\ -200 & 200\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$

For element 3: node 4,2, the finite element equation is

$$k_3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_4\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_4\\ F_2\end{Bmatrix}$$
$$300\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_4\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_4\\ F_2\end{Bmatrix}$$
$$\begin{bmatrix}300 & -300\\ -300 & 300\end{bmatrix}\begin{Bmatrix}u_4\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_4\\ F_2\end{Bmatrix}$$

Assembling element 1, 2 and 3

$$\begin{bmatrix}100 & 0 & -100 & 0\\ 0 & 300 & 0 & -300\\ -100 & 0 & 200+100 & -200\\ 0 & -300 & -200 & 200+300\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$
$$\begin{bmatrix}100 & 0 & -100 & 0\\ 0 & 300 & 0 & -300\\ -100 & 0 & 300 & -200\\ 0 & -300 & -200 & 500\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$

Applying boundary conditions:

At node 1, $u_1 = 0$

At node 2, $u_2 = 0$

Nodal forces, $F_1 = F_2 = F_3 = 0$

$F_4 = 500$ kN

Substitute F1, F2, F3 and F4, u1 and u2 values in assembled matrix

$$\begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

In the above matrix u1 = 0, u2 = 0, so we neglect the first and second row and column of the above matrix. Then the matrix becomes

$$\begin{bmatrix} 300 & -200 \\ -200 & 500 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 500 \end{Bmatrix}$$

$$300u_3 - 200u_4 = 0$$

$$-200\, u_3 + 500\, u_4 = 500$$

Solving the above equation,

$$u_4 = 1.364 \text{ m}; \; u_3 = 0.9091 \text{ m}$$

W.k.t, reaction force, $\{R\} = [k]\{u^*\}-\{F\}$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.9091 \\ 1.364 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 0+0-100(0.9091)+0 \\ 0+0+0-300(1.364) \\ 0+0+300(.9091)-200(1.364) \\ 0+0-200(0.9091)+500(1.364) \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} -90.91 \\ -409.2 \\ 0 \\ 500 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} -90.91 \\ -409.2 \\ 0 \\ 0 \end{bmatrix}$$

R_1= -90.91

R_2 = -409.2

R_3 = 0

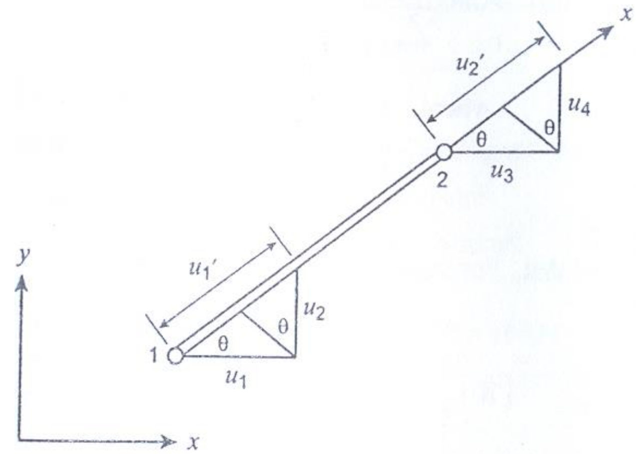
R_4 = 0

W.k.t, reaction force is equivalent and opposite to the applied force.

$R_1 + R_2 + R_3 + R_4$ = -90.91 - 409.2 + 0 + 0 = -500 Kn

SOLID MECHANICS PROBLEMS

18. Derive the stiffness matrix [k] for a truss element.



Consider a two noded bar for the analysis of truss. Truss is subjected to axial forces and axial displacements.

The nodal displacement for this bar element is given by

$$\{u'\} = \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

W.k.t

$u_1' = u_1\cos\theta + u_2\sin\theta$

$u_2' = u_3\cos\theta + u_4\sin\theta$

Let us take l and m as direction cosines, $l = \cos\theta$ and $m = \sin\theta$

$u_1' = u_1 l + u_2 m$

$u_2' = u_3 l + u_4 m$

Changing the above equation into matrix form

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{u'\} = [L]\{u\}$$

$$where\ [\mathrm{L}] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

L is called transformation matrix.

Let us assume (x_1, y_1) and (x_2, y_2) be the co-ordinates of nodes 1 and 2.

We can find l, m and l_e values by using the following formulae.

$l = \cos\theta = \frac{x_2 - x_1}{l_e}$

$m = \sin\theta = \frac{y_2 - y_1}{l_e}$

$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The stiffness matrix for two noded bar element is given by,

$[k'] = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Strain energy, $U = \frac{1}{2}\{u'\}^T [k] \{u'\}$

W. k. t

$\{u'\} = [L] \{u\}$

Substitute $\{u'\}$ in the equation,

$$U = \frac{1}{2}([L]\{u\})^T [k'][L]\{u\}$$

$$U = \frac{1}{2}[L]^T \{u\}^T [k'][L]\{u\}$$

$$U = \frac{1}{2}\{u\}^T \{u\}[k]$$

$$where\ [k] = [L]^T [k'][L]$$

Element stiffness matrix in global co-ordinates

$[k] = [L]^T[k'][L]$

Substitute [L] and [k'] value from equation.

$$[k]=\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\frac{A_eE_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[k]=\frac{A_eE_e}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[k]=\frac{A_eE_e}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} l-0 & m-0 & 0-l & 0-m \\ -l+0 & -m+0 & 0+l & 0+m \end{bmatrix}$$

$$[k]=\frac{A_eE_e}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix}$$

$$[k]=\frac{A_eE_e}{l_e}\begin{bmatrix} l^2-0 & lm-0 & -l^2+0 & -lm+0 \\ lm-0 & m^2-0 & -ml+0 & -m^2+0 \\ 0-l^2 & 0-lm & 0+l^2 & 0+lm \\ 0-lm & 0-m^2 & 0+lm & 0+m^2 \end{bmatrix}$$

$$[k]=\frac{A_eE_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -ml & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

[k] – matrix is symmetric.

Formulation of finite element equation for two noded stress element.

General equation

{F} = [k] {u}

Where, {F} – element force vector

[k] – stiffness matrix

{u} – nodal displacement

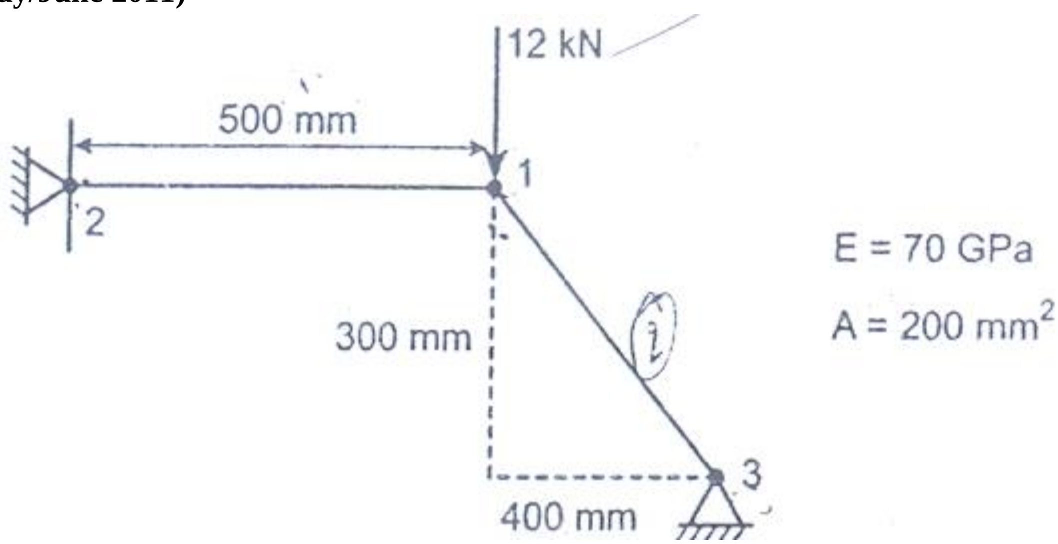
For truss element stiffness matrix is

$$[k] = \frac{A_e E_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -ml & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Substitute [k] value in the general equation,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \frac{A_e E_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -ml & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

19. For the two bar truss in fig. determine the displacement of node 1 and the stress in element 1-3.(May/June 2011)

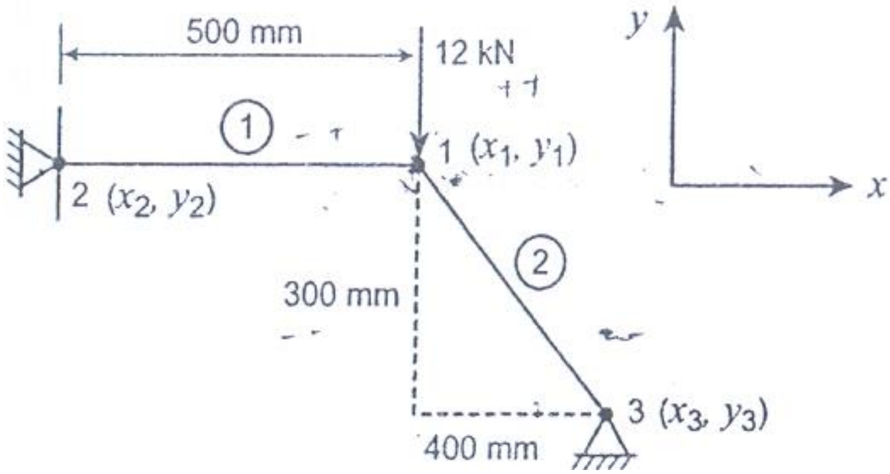


Given:

Young's modulus, E = 70GPa = 70 * 10^3 N/mm^2

Area A = 200 mm^2

Point load at node 1 = 12 kN = 12 * 10^3 N



To find:

a. Displacements of node
b. Stress in element

Solution:

The co-ordinates of various nodes are,

Node 1 = (x_1, y_1) = (0 , 0)

Node 2 = (x_2, y_2) = (-50, 0)

Node 3 = (x_3, y_3) = (400, -300)

For element 1:

Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(-500 - 0)^2 + (0 - 0)^2}$$

l_{e1} = 500 mm

Direction cosines, $l_1 = \frac{x_2 - x_1}{l_{e1}}$

$$= \frac{-500 - 0}{500}$$

l_1 = -1

$$m_1 = \frac{y_2 - y_1}{l_{e1}}$$

$$= \frac{0-0}{500}$$

$$m_1 = 0$$

For element 2:

$$\text{Length, } l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$= \sqrt{(400-0)^2 + (-300-0)^2}$$

$$l_{e2} = 500 \text{ mm}$$

$$\text{Direction cosines, } l_2 = \frac{x_3 - x_1}{l_{e2}}$$

$$= \frac{400}{500}$$

$$l_2 = 0.8$$

$$m_2 = \frac{y_3 - y_1}{l_{e2}}$$

$$= \frac{-300-0}{500}$$

$$m_2 = -0.6$$

For element 1: displacements (u_1, u_2, u_3 and u_4)

Stiffness matrix [k] for truss element is given by

$$[k]_1 = \frac{A_1E_1}{l_{e1}} \begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix}$$

$$[k]_1 = \frac{200*70*10^3}{500}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k]_1 = 28*10^3\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for element 2: displacements (u_1, u_2, u_5 and u_6)

stiffness matrix [k] for truss element is given by

$$[k]_2 = \frac{A_2E_2}{l_{e2}}\begin{bmatrix} l_2^{\,2} & l_2m_2 & -l_2^{\,2} & -l_2m_2 \\ l_2m_2 & m_2^{\,2} & -l_2m_2 & -m_2^{\,2} \\ -l_2^{\,2} & -l_2m_2 & l_2^{\,2} & l_2m_2 \\ -l_2m_2 & -m_2^{\,2} & l_2m_2 & m_2^{\,2} \end{bmatrix}$$

$$[k]_2 = \frac{200*70*10^3}{500}\begin{bmatrix} .64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$[k]_2 = 28*10^3\begin{bmatrix} .64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

Assembling the stiffness matrix [k]

$$[k] = 28*10^3\begin{bmatrix} 1+0.64 & 0+(-0.48) & -1 & 0 & -0.64 & 0.48 \\ 0+(-0.48) & 0+0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix}$$

$$[k] = 28*10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix}$$

W.k.t, General finite element equations is,

{F} = [k] {u}

[k] {u} = {F}

$$28*10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

Applying boundary conditions:

a. Node 2 is fixed, so $u_3 = u_4 = 0$
b. Node 3 is fixed, so $u_5 = u_6 = 0$
c. A point load of $12*10^3$ N is acting at node 1 is downward direction. So, $F_2 = -12*10^3$ N
d. Self weight is neglected. So $F_1 = F_3 = F_4 = F_5 = F_6 = 0$

Substitute the above boundary conditions in the assembled matrix

$$28*10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -12*10^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Neglecting the rows and columns

$$28*10^3\begin{bmatrix}1.64 & -0.48\\ -0.48 & 0.36\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}0\\ -12*10^3\end{Bmatrix}$$

$$28*10^3(1.64u_1-0.48u_2)=0$$

$$28*10^3(-0.48u_1+0.36u_2)=-12*10^3$$

Solving the above equation

u_1 = -0.571mm; u_2 = -1.952 mm

For element 1:

$$\text{Stress } \sigma_1 = \frac{E_1}{l_{e1}}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}$$

For element 2:

$$\text{Stress } \sigma_2 = \frac{E_2}{l_{e2}}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_5\\ u_6\end{Bmatrix}$$

$$=\frac{70*10^3}{500}\begin{bmatrix}-0.8 & 0.6 & 0.8 & -0.6\end{bmatrix}\begin{Bmatrix}-0.571\\ -1.952\\ 0\\ 0\end{Bmatrix}$$

$$=140[(-0.8)*(-0.571)+0.6*(-1.952)+0+0]$$

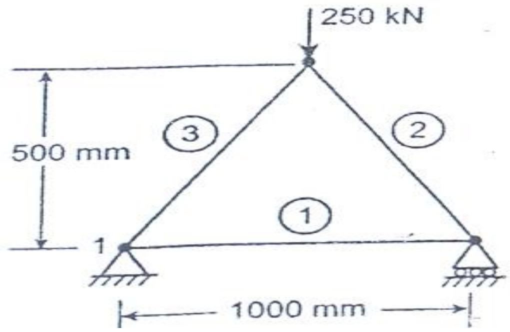
$$=\sigma_2=-100N/mm^2$$

20. Consider a three bar truss as shown in fig. it is given that E=2x10^{-5} N/mm^2 calculate the following;

(i) Nodal displacement

(ii) Stress in each member

(iii) Reactions at the support.



Take : young's modulus, $E=2x10^5$ N/mm^2

Area of element (1)=2000mm^2

Area of element (2)=2500mm^2

Area of element (3)=2500mm^2

To find,

i. Nodal displacements, u_1, u_2, u_3, u_4, u_5 and u_6
ii. Stress in each member, σ_1, σ_2 and σ_3
iii. Reactions at the support R_1, R_2, R_3, R_4, R_5 and R_6

Solution:

The co-ordinates of various nodes are,

Node 1 = (x_1, y_1) = (0 , 0)

Node 2 = (x_2, y_2) = (1000, 0)

Node 3 = (x_3, y_3) = (500, 500)

For element 1:

Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(1000-0)^2 + (0-0)^2}$$

l_{e1} = 1000 mm

Direction cosines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}}$

$= \frac{1000-0}{1000}$

$l_1 = 1$

$m_1 = \frac{y_2 - y_1}{l_{e1}}$

$= \frac{0-0}{1000}$

$m_1 = 0$

For element 2:

Length, $l_{e2} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

$= \sqrt{(500-1000)^2 + (500-0)^2}$

$l_{e2} = 707.107$ mm

Direction cosines, $l_2 = \frac{x_3 - x_2}{l_{e2}}$

$= \frac{500-1000}{707.107}$

$l_2 = -0.707$

$m_2 = \frac{y_3 - y_2}{l_{e2}}$

$= \frac{500-0}{707.107}$

$m_2 = 0.707$

For element 3:

Length, $l_{e3} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$= \sqrt{(500-0)^2 + (500-0)^2}$

l_{e3} = 707.107 mm

Direction cosines, $l_3 = \frac{x_3 - x_1}{l_{e3}}$

$= \frac{500-0}{707.107}$

l_3 = 0.707

$m_3 = \frac{y_3 - y_1}{l_{e3}}$

$= \frac{500-0}{707.107}$

m_2 = 0.707

For element 1: displacements (u_1, u_2, u_3 and u_4)

Stiffness matrix [k] for truss element is given by

$$[k]_1 = \frac{A_1E_1}{l_{e1}} \begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix}$$

$$[k]_1 = \frac{2000*2*10^5}{1000} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k]_1 = 4*10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for element 2: displacements (u_3, u_4, u_5 and u_6)

stiffness matrix [k] for truss element is given by

$$[k]_2 = \frac{A_2E_2}{l_{e2}}\begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_2^2 & -l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_2m_2 \\ -l_2m_2 & -m_2^2 & l_2m_2 & m_2^2 \end{bmatrix}$$

$$[k]_2 = \frac{2500*2*10^5}{707.107}\begin{bmatrix} 0.4998 & -0.4998 & -0.4998 & 0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ 0.4998 & -0.4998 & -0.4998 & 0.4998 \end{bmatrix}$$

$$[k]_2 = 7.07*10^5\begin{bmatrix} 0.4998 & -0.4998 & -0.4998 & 0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ 0.4998 & -0.4998 & -0.4998 & 0.4998 \end{bmatrix}$$

for element 3: displacements (u_1, u_2, u_5 and u_6)

stiffness matrix [k] for truss element is given by

$$[k]_3 = \frac{A_3E_3}{l_{e3}}\begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & -l_3m_3 & -m_3^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}$$

$$[k]_3 = \frac{2500*2*10^5}{707.107}\begin{bmatrix} 0.4998 & -0.4998 & -0.4998 & 0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ 0.4998 & -0.4998 & -0.4998 & 0.4998 \end{bmatrix}$$

$$[k]_3 = 7.07*10^5\begin{bmatrix} 0.4998 & -0.4998 & -0.4998 & 0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ 0.4998 & -0.4998 & -0.4998 & 0.4998 \end{bmatrix}$$

Assembling the stiffness matrix [k]

$$[k]=1*10^5\begin{bmatrix} 4+3.534 & 0+3.534 & -4 & 0 & -3.534 & -3.534 \\ 0+3.534 & 0+3.534 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 4+3.534 & 0+3.534 & -3.534 & 3.534 \\ 0 & 0 & 0-3.534 & 0+3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 3.534+3.534 & -3.534+3.534 \\ -3.534 & -3.534 & 3.534 & -3.534 & -3.534+3.534 & 3.534+3.534 \end{bmatrix}$$

$$[k]=1*10^5\begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ 3.534 & 3.534 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068 \end{bmatrix}$$

W.k.t, General finite element equations is,

{F} = [k] {u}

[k] {u} = {F}

$$1*10^5\begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ 3.534 & 3.534 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

Applying boundary conditions:

a. Node 1 is fixed, so $u_1 = u_2 = 0$
b. Node 2 is moving in x direction so $u_3 \neq 0$ and $u_4 = 0$
c. A point load of $250*10^3$ N is acting in downward direction. So, $F_6 = -250*10^3$ N
d. Self weight is neglected. So $F_1 = F_2 = F_3 = F_4 = F_5 = 0$

Substitute the above boundary conditions in the assembled matrix

$$1*10^5 \begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ 3.534 & 3.534 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ 0 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -250*10^3 \end{Bmatrix}$$

Reducing the above matrix using gauss elimination method

$$\begin{bmatrix} 1 & -0.469 & 0.469 \\ 0 & 5.910 & 1.657 \\ 0 & 0 & 9.902 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2.5 \end{Bmatrix}$$

$$u_3 - 0.469u_5 + 0.469u_6 = 0$$

$$5.910u_5 + 1.657u_6 = 0$$

$$9.902u_6 = -2.5$$

Solving the above equation,

$u_3 = 0.3124$ mm

$u_5 = 0.1562$ mm

$u_6 = -0.5099$ mm

Stress element

$$\sigma = \frac{E}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

For element 1

$$\sigma_1 = \frac{E_1}{l_{e1}}\begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{2*10^5}{1000}\begin{bmatrix}-1 & -0 & 1 & 0\end{bmatrix}\begin{Bmatrix}0\\0\\0.3124\\0\end{Bmatrix}$$

$$= \frac{2*10^5}{1000}*0.3124$$

$$\sigma_1 = 62.48N/mm^2$$

For element 2;

$$\sigma_2 = \frac{E_2}{l_{e2}}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{Bmatrix}u_3\\u_4\\u_5\\u_6\end{Bmatrix}$$

$$= \frac{2*10^5}{707.107}\begin{bmatrix}0.707 & -0.707 & -0.707 & 0.707\end{bmatrix}\begin{Bmatrix}0.31240\\0\\0.1562\\-0.5099\end{Bmatrix}$$

$$= 282.842*(-0.250065)$$

$$\sigma_2 = -70.729N/mm^2$$

For element 3;

$$\sigma_3 = \frac{E_3}{l_{e3}}\begin{bmatrix}-l_3 & -m_3 & l_3 & m_3\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_5\\u_6\end{Bmatrix}$$

$$= \frac{2*10^5}{707.107}\begin{bmatrix}-0.707 & -0.707 & 0.707 & 0.707\end{bmatrix}\begin{Bmatrix}0\\0\\0.1562\\-0.5099\end{Bmatrix}$$

$$= 282.842 * (-0.250065)$$

$$\sigma_3 = -70.729 N/mm^2$$

Reaction force,

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 1*10^5 \begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ 3.534 & 3.534 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.5.34 & -3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 1*10^5 \begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ 3.534 & 3.534 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.5.34 & -3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2.5 \end{Bmatrix}$$

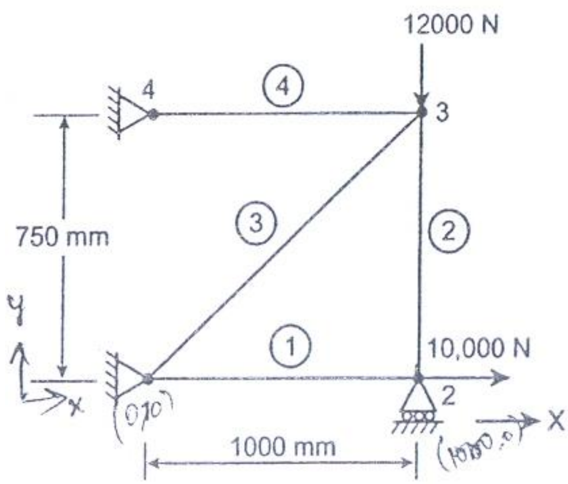
$R_1 = 0$; $R_2 = 1.249*10^5$N; $R_3 = 0$; $R_4 = 1.249*10^5$ N; $R_5 = 0$; $R_6 = 0$

21. Consider a four bar truss as shown in fig. it is given that E=2x10^5 N/mm^2 and A_e=625mm^2 for all elements.

(i) Determine the element stiffness matrix for each element.

(ii) Assemble the structural stiffness matrix K for the entire truss.

(iii) Solve for the nodal displacement.(May/June 2008)

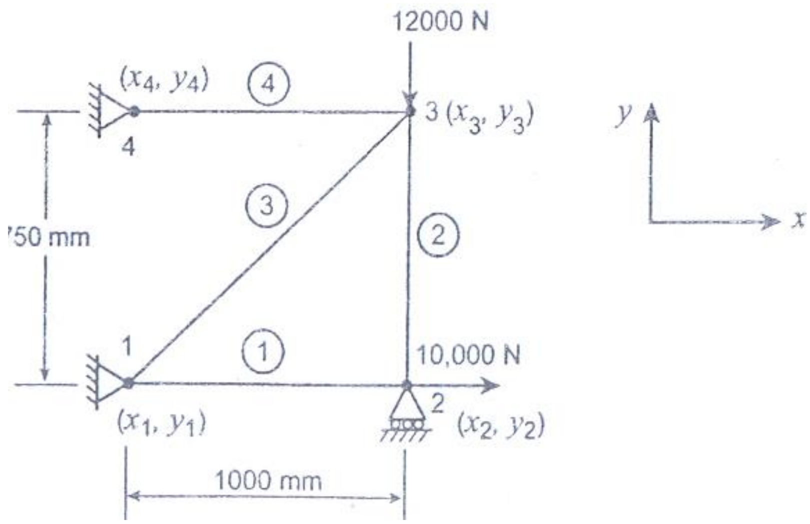


Given: young's modulus, $E=2x10^5$ N/mm^2

Area of each element, $A_e=625mm^2$

Load acting at node 3=-12000N

Load acting node 2 =10,000 N



To find:

(i) Determine the element stiffness matrix for each element.
(ii) Assemble the structural stiffness matrix K for the entire truss.
(iii) Solve sor the nodal displacement.

Solution :consider node 1 as the orgin.

The coordinates of various nodes are given below:

Node 1= (0,0)

Node 2= (1000, 0)

Node 3= (1000,750)

Node 4=(0, 750)

For element (1), length $l_{e1}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$l_{e1}=\sqrt{(1000-0)^2+(0-0)^2}$$

$$l_{e1}=1000mm$$

Direction cosines, $l_1 = \frac{x_2 - x_1}{l_{e1}}$

$l_1=1$

$m_1 = \frac{y_2 - y_1}{l_{e1}}$

$m_1=0$

For element (2): $l_{e1} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

$l_{e1} = \sqrt{(1000 - 1000)^2 + (750 - 0)^2}$

$l_{e2}=750mm$

Direction cosines, $l_2 = \frac{x_3 - x_2}{l_{e2}}$

$l_1=0$

$m_2 = \frac{y_3 - y_2}{l_{e2}} = \frac{750 - 0}{750}$

$m_2=1$

For element (3): length, $l_{e3} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = \sqrt{(1000 - 1000)^2 + (750 - 0)^2}$

$l_{e3}=1250mm$

Direction cosines, $l_3 = \frac{x_3 - x_1}{l_{e3}} = \frac{1000 - 0}{1250}$

$l_3=0.8$

$m_3 = \frac{y_3 - y_1}{l_{e3}} = \frac{750 - 0}{1250}$

$m_3=0.6$

for element 4: length $l_{e4} = \sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2} = \sqrt{(1000 - 0)^2 + (750 - 750)^2}$

$l_{e4}=1000mm$

Direction cosines, $l_4 = \frac{x_3 - x_4}{l_{e4}} = \frac{1000-0}{1000}$

$l_4 = 1$

$m_4 = \frac{y_3 - y_4}{l_{e4}} = \frac{750-750}{1000}$

$m_4 = 0$

For element (1): displacement u_1, u_2, u_3 and u_4)

Stiffness matrix [k] for a truss element is given by,

$$[k] = \frac{A_1 E_1}{l_{e1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & -m_1^2 \end{bmatrix}$$

$$= \frac{625x2x10^5}{1000} \begin{bmatrix} (1)^2 & 0 & -(1)^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(1)^2 & 0 & (1)^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 1.25x10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k]_1 = 1x10^5 \begin{bmatrix} 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For element (2); displacement u_3, u_4 u_5 and $u_{6)}$;

Stiffness matrix, $[k]_2 = \frac{A_2 E_2}{l_{e2}} \begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_2^2 & l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_2 m_2 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & -m_2^2 \end{bmatrix}$

$$= \frac{625x2x10^5}{750}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$=1.66x10^5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k]_2=1x10^5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.66 & 0 & -1.66 \\ 0 & 0 & 0 & 0 \\ 0 & -1.66 & 0 & 1.66 \end{bmatrix}$$

For element (3): (displacement u_1, u_2 u_5 and u_6)

$$\text{Stiffness matrix, } [k]_3=\frac{A_3E_3}{l_{e3}}\begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & l_3m_3 & -m_3^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & -m_3^2 \end{bmatrix}$$

$$= \frac{625x2x10^5}{1250}\begin{bmatrix} (0.8)^2 & 0.8x0.6 & (-0.8)^2 & -0.8x0.6 \\ 0.8x0.6 & (0.6)^2 & -0.8x0.6 & (-0.6)^2 \\ (-0.8)^2 & -0.8x0.6 & (0.8)^2 & 0.8x0.6 \\ -0.8x0.6 & (-0.6)^2 & 0.8x0.6 & (0.6)^2 \end{bmatrix}$$

$$[k]_3=1x10^5\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

For element (4): displacement u_7, u_8, u_5 and u_6)

$$\text{Stiffness matrix, } [k]_4=\frac{A_4E_4}{l_{e4}}\begin{bmatrix} l_4^2 & l_4m_4 & -l_4^2 & -l_4m_4 \\ l_4m_4 & m_4^2 & l_4m_4 & -m_4^2 \\ -l_4^2 & -l_4m_4 & l_4^2 & l_4m_4 \\ -l_4m_4 & -m_4^2 & l_4m_4 & m_4^2 \end{bmatrix}$$

$$=\frac{625x2x10^5}{1000}\begin{bmatrix}(1)^2 & 0 & -(1)^2 & 0\\ 0 & 0 & 0 & 0\\ -(1)^2 & 0 & (1)^2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$=1.25x10^5\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$[k]_4=1x10^5\begin{bmatrix}1.25 & 0 & -1.25 & 0\\ 0 & 0 & 0 & 0\\ -1.25 & 0 & 1.25 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Assemble the stiffness matrix[k], assemble the equation (1) (2) (3) and (4)

$$[k]=1x10^5\begin{bmatrix}1.25+0.64 & 0+0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0\\ & 0+0.36 & 0 & 0 & -0.48 & -0.36 & 0 & 0\\ & & 1.25+0 & 0 & 0 & 0 & 0 & 0\\ & & & 0+1.666 & 0 & -1.666 & 0 & 0\\ & \boxed{\text{Symmetric matrix}} & & & 0+0.64+1.25 & 0+0.48+0 & -1.25 & 0\\ & & & & & 1.666+0.36+0 & 0 & 0\\ & & & & & & 1.25 & 0\\ & & & & & & & 0\end{bmatrix}$$

$$[k]=1x10^5\begin{bmatrix}1.89 & 0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0\\ 0.48 & 0.36 & 0 & 0 & -0.48 & 0.36 & 0 & 0\\ -1.25 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.666 & 0 & -1.666 & 0 & 0\\ -0.64 & -0.48 & 0 & 0 & 1.89 & 0.48 & -1.25 & 0\\ -0.48 & -0.36 & 0 & -1.666 & 0.48 & 2.026 & 0 & 0\\ 0 & 0 & 0 & 0 & -1.25 & 0 & 1.25 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

we know that, general finite element equation is,

{F} =[k] {u}

[k] {u}={F}

$$1x10^5 \begin{bmatrix} 1.89 & 0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0 \\ 0.48 & 0.36 & 0 & 0 & -0.48 & 0.36 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.666 & 0 & -1.666 & 0 & 0 \\ -0.64 & -0.48 & 0 & 0 & 1.89 & 0.48 & -1.25 & 0 \\ -0.48 & -0.36 & 0 & -1.666 & 0.48 & 2.026 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Applying boundary conditions conditions: refer fig:

1. Node 1 is fixed. So, $u_1=u_2=0$
2. Node 4 is fixed. So, $u_7=u_8=0$
3. Node 2 is moving in x direction. So $u_3 \neq 0$ and $u_4=0$
4. At node 3, point load of 12,000N is acting in downward direction. So, F_6=-12,000N.
5. At node 2, point load of 10,000 N is acting x direction. So, F_3=10,000N.
6. Self-weight is neglected. So, $F_1=F_2=F_3=F4=F_5=F_6=F_7=F_8=0$

Substitute boundary condition values in equation, $u_1 = u_2 = u_4 = u_7 = u_8 = 0$

The final reduced equation is

$$1*10^5 \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1.89 & 0.48 \\ 0 & 0.48 & 2.026 \end{bmatrix} \begin{Bmatrix} u3 \\ u5 \\ u6 \end{Bmatrix} = \begin{Bmatrix} 10000 \\ 0 \\ -12000 \end{Bmatrix}$$

Reduce the above matrix using gauss elimination method,

$$1*10^5 \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1.89 & 0.48 \\ 0 & 0 & 1.9041 \end{bmatrix} \begin{Bmatrix} u3 \\ u5 \\ u6 \end{Bmatrix} = \begin{Bmatrix} 8000 \\ 0 \\ -12000 \end{Bmatrix}$$

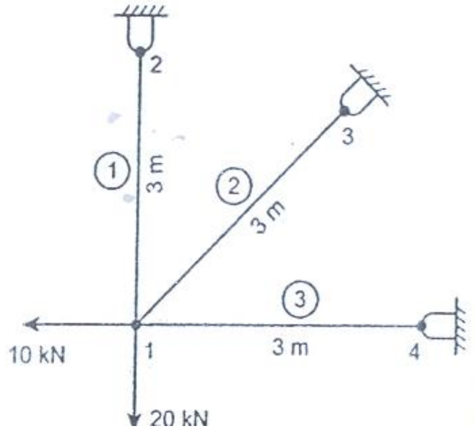
$1*10^5(1.9041\ u_6) = -12000$

$1*10^5(1.89u_5 + 0.48u_6) = 0$

$1*10^5(1.25u_3) = 8000$

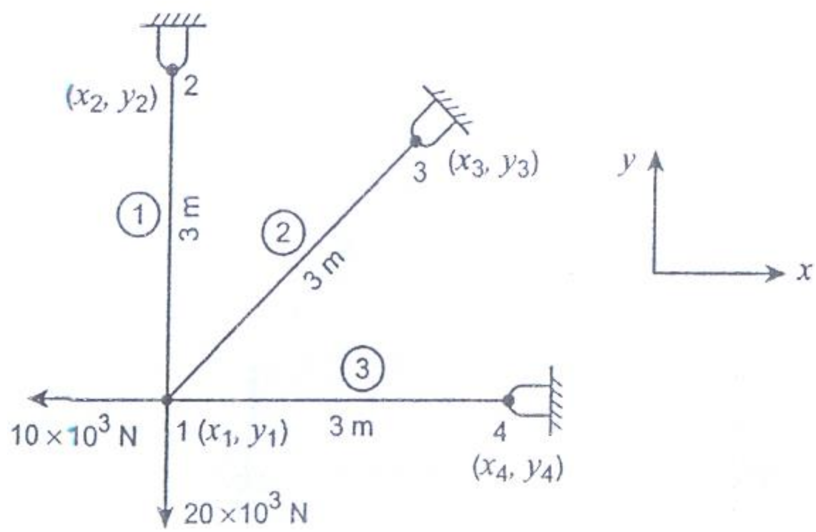
$u_3 = 0.08$mm; $u_5 = 0.016$ mm; $u_6 = -0.063$ mm

22. For the plane truss shown in fig. determine the horizontal and vertical displacement of nodal and the stresses in each element, all element have E=201Gpa and A=4x10^{-4}m^2.



Given:

Young's modulus E = 201 Gpa = $201*10^9$ N/m^2
Area of each element, A = $4*10^{-4}$m^2
Load acting, l = $-20*10^3$ N
Load acting, 2 = $-10*10^3$N



To find:

i. Nodal displacements, u_1, u_2, u_3, u_4, u_5, u_6, u_7 and u_8
ii. Stress in each element, σ_1, σ_2 and σ_3.

Solution: consider node 1 as the origin.

The co – ordinates of various nodes are given below

Node 1 = (x_1, y_1) = (0, 0)

Node 2 = (x_2, y_2) = (0, 3)

Node 4 = (x_4, y_4) = (3, 0)

Node 3 = (x_3, y_3) = (3*cos 450, 3* sin 450) = (2.121m, 2.121m)

For element (1), length $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$l_{e1} = \sqrt{(0-0)^2 + (3-0)^2}$$

$$l_{e1} = 3m$$

Direction cosines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}}$

$$l_1 = 0$$

$$m_1 = \dfrac{y_2 - y_1}{l_{e1}}$$

$$m_1 = 3$$

For element (2): $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$l_{e2} = \sqrt{(2.121-0)^2 + (2.121-0)^2}$$

$$l_{e2} = 3m$$

Direction cosines, $l_2 = \dfrac{x_3 - x_1}{l_{e2}}$

$$l_1 = 0.707$$

$$m_2 = \dfrac{y_3 - y_1}{l_{e2}} = \dfrac{2.121 - 0}{3}$$

$$m_2 = 0.707$$

For element (3): length, $l_{e3} = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2} = \sqrt{(3-0)^2 + (0-0)^2}$

$$l_{e3} = 3m$$

Direction cosines, $l_3 = \frac{x_4 - x_1}{l_{e3}} = \frac{3-0}{3}$

$l_3=1$

$m_3 = \frac{y_4 - y_1}{l_{e3}} = \frac{0-0}{3}$

$m_3=0$

For element (1): displacement u_1,u_2, u_3 and u_4)

Stiffness matrix [k] for a truss element is given by,

$$[k]=\frac{A_1E_1}{l_{e1}}\begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & -m_1^2 \end{bmatrix}$$

$$= \frac{4*10^{-4}*201*10^9}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$=268x10^5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For element (2); displacement u_1, u_2 u_5 and u_6);

Stiffness matrix, $[k]_2=\frac{A_2E_2}{l_{e2}}\begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_2^2 & l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_2m_2 \\ -l_2m_2 & -m_2^2 & l_2m_2 & -m_2^2 \end{bmatrix}$

$$= \frac{4*10^{-4}*201*10^9}{3}\begin{bmatrix} -0.707^2 & 0.707*0.707 & -0.707^2 & -0.707*0.707 \\ 0.707*0.707 & 0.707^2 & -0.707*0.707 & -0.707^2 \\ -0.707^2 & -0.707*0.707 & 0.707^2 & 0.707*0.707 \\ -0.707*0.707 & -0.707^2 & 0.707*0.707 & 0.707^2 \end{bmatrix}$$

$$[k]_2=268x10^5\begin{bmatrix} 0.499 & 0.499 & -0.499 & -0.499 \\ 0.499 & 0.499 & -0.499 & -0.499 \\ -0.499 & -0.499 & 0.499 & 0.499 \\ -0.499 & -0.499 & 0.499 & 0.499 \end{bmatrix}$$

For element (3): (displacement u_1, u_2 u_7 and u_8)

$$\text{Stiffness matrix, } [k]_3=\frac{A_3E_3}{l_{e3}}\begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & l_3m_3 & -m_3^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & -m_3^2 \end{bmatrix}$$

$$=\frac{4*10^{-4}*201*10^9}{3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k]_3=268x10^5\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Assembling the above matrix

$$[k]=268*10^5\begin{bmatrix} 1.499 & 0.499 & 0 & 0 & -0.499 & -0.499 & -1 & 0 \\ 0.499 & 1.499 & 0 & -1 & -0.499 & -0.499 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

General finite element equation,

$$[k] = 268*10^5 \begin{bmatrix} 1.499 & 0.499 & 0 & 0 & -0.499 & -0.499 & -1 & 0 \\ 0.499 & 1.499 & 0 & -1 & -0.499 & -0.499 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Applying boundary conditions

- Node 2 is fixed, so $u_3 = u_4 = 0$
- Node 3 is fixed, $u_5 = u_6 = 0$
- Node 4 is fixed, so $u_7 = u = 0$
- At node 1 a point load of $20*10^3$N is acting opposite to x direction, so $F_1 = -10*10^3$ N.
- Self weight is neglected, so $F_3 = F_4 = F_5 = F_6 = F_7 = F_8 = 0$

The final reduced equation is

$$268*10^5 \begin{bmatrix} 1.499 & 0.499 \\ 0.499 & 1.499 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$268*10^5 \begin{bmatrix} 1.499 & 0.499 \\ 0.499 & 1.499 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -10*10^3 \\ -20*10^3 \end{Bmatrix}$$

$268*10^5\ (1.499u_1+0.499u_2) = -10*10^3$

$268*10^5(0.499u_1 + 1.499u_2) = -20*10^3$

$u_1 = -9.379*10^{-5}$ m, $u_2 = -4.66 * 10^{-4}$ m

Stress element

$$\sigma = \frac{E}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

For element 1

$$\sigma_1 = \frac{E_1}{l_{e1}} \begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{201*10^9}{3} \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -9.379*10^{-5} \\ -4.66*10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma_1 = 31.22*10^6 \, N/m^2$$

For element 2;

$$\sigma_2 = \frac{E_2}{l_{e2}} \begin{bmatrix} -l_2 & -m_2 & l_2 & m_2 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= \frac{201*10^9}{3} \begin{bmatrix} -0.707 & -0.707 & 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} -9.379*10^{-5} \\ -4.66*10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma_2 = 26.52*10^6 \, N/m^2$$

For element 3;

$$\sigma_3 = \frac{E_3}{l_{e3}} \begin{bmatrix} -l_3 & -m_3 & l_3 & m_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_7 \\ u_8 \end{Bmatrix}$$

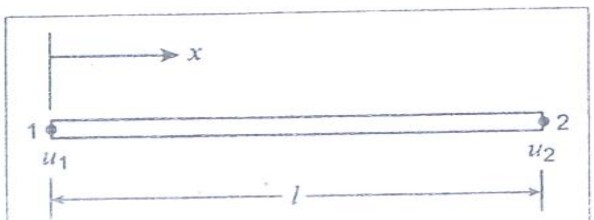
$$= \frac{201*10^9}{3}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}-9.379*10^{-5}\\ -4.66*10^{-4}\\ 0\\ 0\end{Bmatrix}$$

$$\sigma_3 = 6.28*10^6\, N/m^2$$

HEAT TRANSFER

23. Derive the stiffness matrix for one dimensional heat conduction element. (May/June 2013)

Consider a bar element with nodes 1 and 2 as shown in Fig. T1 and T2 are the Temperatures at the respective nodes. Let k be the thermal conductivity of the material.
Let l be the length of the bar eleemnt



Stiffness matrix

W.K.T, stiffness matrix [k] = $\int_v [B]^T [D][B] dv$

In 1'D bar element,

Temperature function, T = N1T1 + N2T2

Where, $N_1 = \frac{1-x}{l};\ \mathrm{N}_2 = \frac{x}{l}$

W.K.T Strain displacement,

$$[B] = \begin{bmatrix}\frac{dN_1}{dx} & \frac{dN_2}{dx}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{-1}{l} & \frac{1}{l}\end{bmatrix}$$

$$[B]^T = \begin{bmatrix}\frac{dN_1}{dx}\\ \frac{dN_2}{dx}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{-1}{l}\\ \frac{1}{l}\end{bmatrix}$$

In one dimensional problems, [D] = k= Thermal conductivity of the material.

Substitute $[B]^T[D][B]$ value stiffness matrix

Stiffness matrix for heat conduction,

$$[k_c] = \int_0^1 \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} * k * \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$[k] = \int_0^1 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} kAdx \qquad \because \text{dv} = \text{Adx}$$

$$[k] = kA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^1 dx$$

$$[k] = kA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$[k] = kA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [l]$$

$$\boxed{[k_c] = \frac{Ak}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Where,

A= Area of the element, m^2

k= thermal conductivity of the element, W/mK,

l= length of the element, m

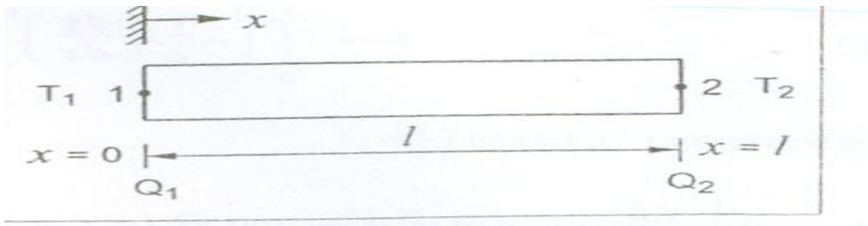
Nodal force vector due to increase in temperature in the member is given by,

$$\text{Heat transfer } \{F\} = EA\alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

24. Derive the equation for one dimensional heat transfer element based on the stationary of a functional.

Consider a bar element with nodes 1 and 2 as shown in fig. T_1 and T_2 are the temperatures at the respective nodes, so T_1 and T_2 are considered as degrees of freedom of this bar element.

We know that, $T(x)= N_1T_1+N_2T_2$



$T(x)= (1-\frac{x}{l})\ T_1+\ \frac{x}{l}T_2$

here $N_1=1-x/l$; $N_2=x/l$

The strain energy stored within the element is given by,

$$U=\frac{1}{2}\int_0^1 k\ (\frac{dT}{dx})^2\ dx$$

Potential energy of external force is given by

$$H=\int_0^1 q_0 T\ dx+Q_1T_1+Q_2\ T_2$$

The total potential energy,

$$\pi=U-H$$

$$=\frac{1}{2}\int_0^1 k\left(\frac{dT}{dx}\right)^2 dx-\int_0^1 qTdx-Q_1T_1-Q_2T_2$$

Differentiating the eqn T(x),

$$\frac{dT}{dx}=-\frac{1}{l}T_1+\frac{1}{l}T_2$$

$$\frac{dT}{dx}=\frac{1}{l}(T_2-T_1)$$

Substitute the equation

$$,\ \pi=\frac{1}{2}\int_0^l k\ [\frac{1}{l}(T_2-\ T_1)]^2\ dx-\int_0^l q_0\ T\ dx - QT_1-QT_2.$$

Integrating the above eqn

$$\pi = k/2l[T_2^2 + T_1^2 - 2T_1T_2] - [q_0T_l] - QT_1 - QT_2$$

$$\frac{k}{l}\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\ \{\begin{matrix} T_1 \\ T_2 \end{matrix}\} = \{\begin{matrix} \frac{q0\ l}{2} \\ \frac{q0\ l}{2} \end{matrix}\}+\{\begin{matrix} Q_1 \\ Q_2 \end{matrix}\}$$

$[k_c]\ \{T\}=\{F\}.$

HEAT TRANSFER PROBLEM

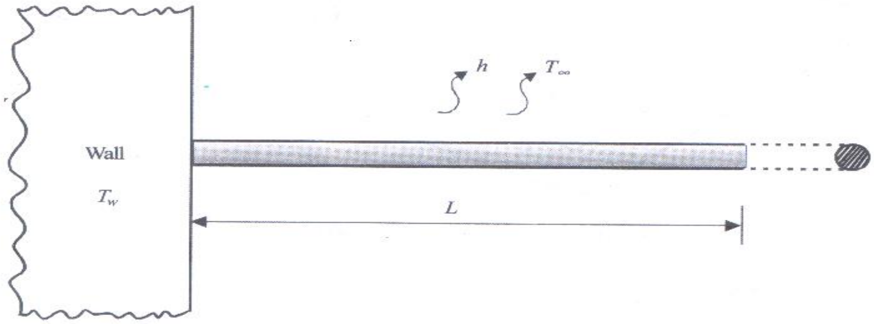
25. Consider a 1mm diameter, 50 mm long aluminum pin-fin as shown fig. used to enhance the heat transfer from a surface wall maintained at 300˚C. the governing differential equation and the boundary conditions are given by

$$k\frac{d^2T}{dx^2}=\frac{p\,h}{A}(T-T_\infty)$$

$$T(0)=T_w=300^\circ C$$

$$\frac{d^2T}{dx^2}L=0 \text{ (insulated tip)}$$

Where, k=thermal conductivity, p= perimeter, A= cross-sectional area, h= convective heat transfer coefficient, T_w=wall temperature, T_∞=ambient temperature.Let, k=200W/m˚C for aluminum, h=20 W/m^2˚C, T_∞=30˚C.esmate the temperature distribution in the fin using the galerkin weighted residual method.



Given:

Diameter, d=1mm=1x10^{-3}

Length , l= 50mm= 50x10^{-3}m

Wall temperature, T_w=300˚C

Governing different equation, $K\frac{d^2T}{dx^2}=\frac{p\,h}{A}(T-T_\infty)$

$$T(0)=T_w=300^\circ C$$

$$\frac{dT}{dx}(L)=0$$

Thermal conductivity, k=200w/m˚C.

Heat transfer co-efficient h = 20 W/m$^{2 0}$C

Ambient temperature, T_∞ = 30^{0}C

To find: Temperature distribution using Galerkin's method.

Solution:

Assume a trail solution. Let.

$$T(x)=a_0+a_1x+a_2x^2$$

The boundary conditions are, $T(0)=T_w=300^\circ C$

$$\frac{dT}{dx}(L)=0$$

From equation (a), x=0, T=300˚C

Applying these values in equation (1),

$$300=a_0$$

$$a_0=300$$

From equation (b), x=L, $\frac{dT}{dx}$=0

Differential equation (1), $\frac{dT}{dx}=a_1+a_2.2x$ -------------2

$$0=a_1+a_2(2L)$$

$$a_1 = -2La_2$$

Substitute a_0 and a_1 values in equation (1)

$$T(x)=a_0+a_1x+a_2x^2$$

$$T(x) = 300 + (-2a_2L)x+a_2x^2$$

$$T(x) = 300 + a_2(x^2-2Lx) \text{------- } 3$$

w.k.t, $kd^2T/dx^2 = Ph/A\ (T - T_\infty)$

$200\ d^2T/dx^2 = \pi(1*10^{-3})*20/(\pi/4)(1*10^{-3})^2\ \ (T – 30)$ [here p=πD]

$$d^2T/dx^2 = 400\ (T – 30) \text{ - - - - - } 4$$

substitute T value from eqn 3

$$d^2T/dx^2 = 400\ (300 + a_2(x^2-2Lx)-30)$$

$$=400[270+a_2(x^2-2Lx)] \text{ - - - } 5$$

From eqn 2, $dT/dx = a_1+a_2(2x)$

$$d^2T/dx^2 = 2a_2 \text{ - - - - - } 6$$

substitute d^2T/dx^2value in equation 5

$$2a_2 = 400[270+a_2(x^2-2Lx)]$$

$$2a_2 - 400[270+a_2(x^2-2Lx)] = 0$$

Take residual, $R=2a_2-400[270+a_2(x^2-2Lx)$

$W(x) = x^2-2Lx$

$$\int_0^l \left(x^2-2Lx\right)R\,dx = 0$$

$$\int_0^L \left(x^2-2Lx\right)\left[2a_2-400\left(270+a_2(x^2-2Lx)\right)\right]dx = 0$$

$$\int_0^L \left(x^2-2Lx\right)\left[2a_2-108000-400a_2x^2+800a_2Lx\right]dx = 0$$

$$\int_0^L \left[2a_2x^2-108000x^2-400a_2x^4+800a_2Lx^3-4a_2Lx+216000Lx+800a_2Lx^3-1600a_2L^2x^2\right]dx = 0$$

$$\left[2a_2\frac{x^3}{3}-108000\frac{x^3}{3}-400a_2\frac{x^5}{5}+800a_2L\frac{x^4}{4}-4a_2L\frac{x^2}{2}+216000L\frac{x^2}{2}+800a_2L\frac{x^4}{4}-1600a_2L^2\frac{x^3}{3}\right]_0^L = 0$$

$$\left[2a_2\frac{L^3}{3}-108000\frac{L^3}{3}-400a_2\frac{L^5}{5}+800a_2L\frac{L^4}{4}-4a_2L\frac{L^2}{2}+216000L\frac{L^2}{2}+800a_2L\frac{L^4}{4}-1600a_2L^2\frac{L^3}{3}\right] = 0$$

$$\left[2a_2\frac{L^3}{3}-108000\frac{L^3}{3}-400a_2\frac{L^5}{5}+800a_2\frac{L^5}{4}-4a_2\frac{L^3}{2}+216000\frac{L^3}{2}+800a_2\frac{L^5}{4}-1600a_2\frac{L^5}{3}\right] = 0$$

$$\left[\frac{2a_2}{3}-\frac{108000}{3}-\frac{400a_2L^2}{5}+\frac{800a_2L^2}{4}-\frac{4a_2}{2}+\frac{216000}{2}+\frac{800a_2L^2}{4}-\frac{1600a_2L^2}{3}\right] = 0$$

$$a_2\left[0.667-80L^2+200L^2-2+200L^2-533.33L^2\right] = -72000$$

$$L = 50*10^{-3}\,m(given)$$

$$a_2\left[0.667-0.2+0.5-2+0.5-1.33\right] = -72000$$

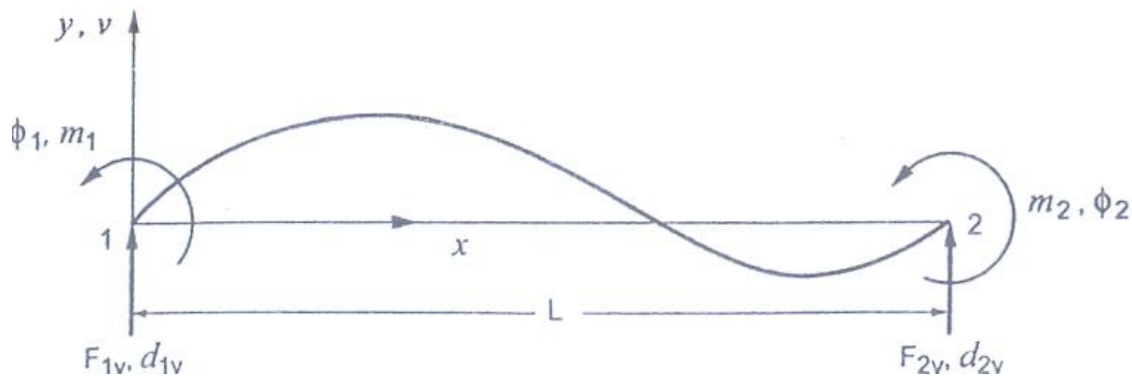
$$a_2 = 38572.8$$

Galerkin solution, $T(x)=300+38572.80(x^2-2Lx)$

FOURTH ORDER BEAM EQUATION
BEAM ELEMENT AND EQUATIONS:

26. Derive the shape function for beam element.

Consider the beam element of length L with axial local co-ordinate x and transverse local co-ordinate y. the local transverse nodal displacements are given by d_{1y} and d_{2y}. the rotations are given by ϕ_1 and ϕ_2. The local nodal forces are given by F_{1y} and F_{2y}. the bending moments are given by m_1 and m_2.



Sign conversion for all nodes

i. Moments are positive in the counterclockwise direction.
ii. Rotations are positive in the counterclockwise direction.
iii. Forces are positive in the positive y direction.
iv. Displacements are positive in the positive y direction.
v.

Assume transverse displacement variation through the element length to be

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

we express v in terms of the nodal degrees of freedom d_{1y}, d_{2y}, ϕ_1 and ϕ_2 as follows:

At x = 0,

$$v(0) = a_4 = d_{1y}$$

$$\frac{dv(x)}{dx} = 3a_1x^2 + 2a_2x + a_3$$
$$\frac{dv(0)}{dx} = a_3 = \phi_1$$

When, x = L,

$$v(L) = a_1L^3 + a_2L^2 + a_3L + a_4 = d_{2y}$$

$$\frac{dv(x)}{dx} = 3a_1x^2 + 2a_2x + a_3$$
$$\frac{dv(0)}{dx} = 3a_1L^2 + 2a_2L + a_3 = \phi_2$$

Where, $\phi = \frac{dv}{dx}$

Finding a_1 and a_2 in terms of d_{1y}, d_{2y}, ϕ_1 and ϕ_2 by using the above equation.

$$a_1L^3 + a_2L^2 + a_3L + a_4 = d_{2y}$$
$$a_1L^3 + a_2L^2 + a_3L + d_{1y} = d_{2y}$$
$$(d_{2y} - d_{1y}) = a_1L^3 + a_2L^2 + a_3L$$
$$(d_{2y} - d_{1y}) = a_1L^3 + a_2L^2 + \phi_1 L$$
$$(d_{2y} - d_{1y}) - \phi_1 L = a_1L^3 + a_2L^2$$
$$\frac{1}{L}(d_{2y} - d_{1y} - \phi_1 L) = a_1L^2 + a_2L \text{ - - - - - } 1$$
$$3a_1L^2 + 2a_2L + a_3 = \phi_2$$
$$3a_1L^2 + 2a_2L + \phi_1 = \phi_2$$
$$3a_1L^2 + 2a_2L = \phi_2 - \phi_1 \text{ - - - - - - - - - - } 2$$

Solving the equation 1 and 2

$$a_2 = \frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)$$
$$a_1 = \frac{2}{L^3}(d_{1y} - d_{2y}) - \frac{1}{L^2}(\phi_1 + \phi_2)$$

Substitute a_1, a_2, a_3 and a_4 value in the equation $v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$

$$v(x) = \frac{2}{L^3}(d_{1y} - d_{2y}) - \frac{1}{L^2}(\phi_1 + \phi_2)x^3 + \frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)x^2 + \phi_1 x + d_{1y}$$

Convert the above equation into matrix form,

$$v(x) = [N]\{d\}$$

$$v(x) = [N_1\ N_2\ N_3\ N_4] \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$v(x) = N_1 d_{1y} + N_2 \phi_1 + N_3 d_{2y} + N_4 \phi_2$$

Where N_1, N_2, N_3 and N_4 are shape function for beam element.

$$N_1 = \frac{1}{L^3}\left(2x^3 - 3x^2 L + L^3\right)$$

$$N_2 = \frac{1}{L^3}\left(x^3 L - 2x^2 L^2 + xL^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2x^3 + 3x^2 L\right)$$

$$N_4 = \frac{1}{L^3}\left(x^3 L - x^2 L^2\right)$$

27. Derive the stiffness matrix [k] for beam element.

The stiffness matrix for beam element is derived using equilibrium approach and beam theory sign conversions.

We know that

Transverse displacement

$$v(x) = \left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right]x^3 + \left[\frac{-3}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L}\left(2\phi_1 + \phi_2\right)\right]x^2 + \phi_1 x + d_{1y}$$

$$\frac{dv(x)}{dx} = 3x^2\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right] + 2x\left[\frac{-3}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L}\left(2\phi_1 + \phi_2\right)\right] + \phi_1$$

$$\frac{d^2v(x)}{dx^2} = 6x\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right] + 2\left[\frac{-3}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L}\left(2\phi_1 + \phi_2\right)\right]$$

$$\frac{d^3v(x)}{dx^3} = 6\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right]$$

put x = 0 in equation $\frac{d^2v(x)}{dx^2}$

$$\frac{d^2v(0)}{dx^2} = 0 + 2\left[\frac{-3}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L}\left(2\phi_1 + \phi_2\right)\right]$$

$$= \frac{-6}{L^2}\left(d_{1y} - d_{2y}\right) - \frac{2}{L}\left(2\phi_1 + \phi_2\right)$$

$$= \frac{1}{L^3}\left[-6Ld_{1y} + 6Ld_{2y} - 4L^2\phi_1 - 2L^2\phi_2\right]$$

$$\frac{d^2v(0)}{dx^2} = \frac{1}{L^3}\left[-6Ld_{1y} + 6Ld_{2y} - 4L^2\phi_1 - 2L^2\phi_2\right]$$

put x = L in equation $\frac{d^2v(x)}{dx^2}$

$$\frac{d^2v(L)}{dx^2} = 6L\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right] + 2\left[\frac{-3}{L2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L}\left(2\phi_1 + \phi_2\right)\right]$$

$$= \frac{12L}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{6L}{L^2}\left(\phi_1 + \phi_2\right) - \frac{6}{L^2}\left(d_{1y} - d_{2y}\right) - \frac{2}{L}\left(2\phi_1 + \phi_2\right)$$

$$\frac{d^2v(L)}{dx^2} = \frac{1}{L^3}\left[6Ld_{1y} - 6Ld_{2y} + 2L^2\phi_1 + 4L^2\phi_2\right]$$

put x = 0 in equation $\frac{d^3v(x)}{dx^3}$

$$\frac{d^3v(0)}{dx^3} = 6\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right]$$

$$= \frac{1}{L^3}\left[12d_{1y} - 12d_{2y} + 6L\phi_1 + 6L\phi_2\right]$$

put x = L in equation $\frac{d^3v(x)}{dx^3}$

$$\frac{d^3v(L)}{dx^3} = 6\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}\left(\phi_1 + \phi_2\right)\right]$$

$$= \frac{1}{L^3}\left[12d_{1y} - 12d_{2y} + 6L\phi_1 + 6L\phi_2\right]$$

We know that

Nodal force, $F_{1y} = EI\frac{d^3v(0)}{dx^3}$

$$F_{1y} = \frac{EI}{L^3}\left[12d_{1y} - 12d_{2y} + 6L\phi_1 + 6L\phi_2\right]$$

Bending moment, m_1 = -EI$\frac{d^2v(0)}{dx^2}$

$$= \frac{-EI}{L^3}\left[-6Ld_{1y} + 6Ld_{2y} - 4L^2\phi_1 - 2L^2\phi_2\right]$$

Nodal force, F_{2y} = -EI$\frac{d^3v(L)}{dx^3}$

$$F_{2y} = -\frac{EI}{L^3}\left[12d_{1y} - 12d_{2y} + 6L\phi_1 + 6L\phi_2\right]$$

$$F_{2y} = \frac{EI}{L^3}\left[-12d_{1y} + 12d_{2y} - 6L\phi_1 - 6L\phi_2\right]$$

Bending moment, m_2 = EI$\frac{d^2v(L)}{dx^2}$

$$= \frac{EI}{L^3}\left[6Ld_{1y} - 6Ld_{2y} + 2L^2\phi_1 + 4L^2\phi_2\right]$$

Arranging to the above equation (F1y, m1, F2y, m2) in matrix form

$$\begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

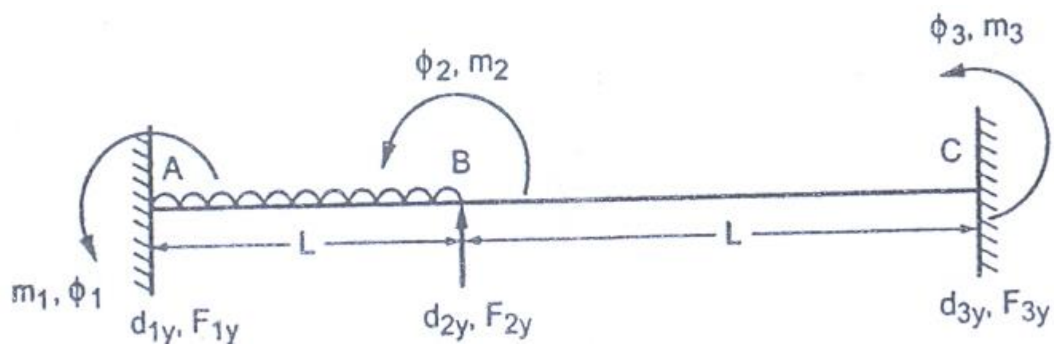
Finite element equation for a beam element

Stiffness matrix, $[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

28. A fixed beam of length 2l m carries a uniformly distributed load of w(N/m) which run over a length L m from the fixed end shown in fig.

(i) Calculate the rotation at point B.

Given:



Young's modulus, E

Moment of inertia, I

Element 1, length $L_1 = L$

` element 2, length $L_2 = L$

To find

i. Slope or rotation at B, ϕ_2

Solution:

We can divide the beam into two element

For element 1: node 1, 2, 3 and 4; d_{1y}, ϕ_1, d_{2y}, ϕ_2)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix}$$

For uniformly distributed load

$F_{1y} = -WL/2$, $m_1 = -WL^2/12$, $F_{2y} = -WL/2$, $m_2 = WL^2/12$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} \frac{-WL}{2} \\ \frac{-WL^2}{12} \\ \frac{-WL}{2} \\ \frac{WL^2}{12} \end{Bmatrix}$$

For element 2: node 3,4,5 and 6; d_{2y}, ϕ_2, d_{3y}, ϕ_3)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix}$$

For uniformly distributed load,

$F_{2y} = 0$, $m_2 = 0$, $F_{3y} = 0$, $m_3 = 0$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

assemble the finite element

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-WL}{2} \\ \dfrac{-WL^2}{12} \\ \dfrac{-WL}{2}+0 \\ \dfrac{WL^2}{12}+0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-WL}{2} \\ \dfrac{-WL^2}{12} \\ \dfrac{-WL}{2}+0 \\ \dfrac{WL^2}{12}+0 \\ 0 \\ 0 \end{Bmatrix}$$

Applying boundary condition;

- A is fixed, displacement d_{1y} and rotation ϕ_1 are 0
- At B, vertical displacement, $d_{2y} = 0$
- At C, vertical displacement, $d_{3y} = 0$

Substitute the boundary condition in the global stiffness matrix,

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \phi_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{-WL}{2} \\ \frac{-WL^2}{12} \\ \frac{-WL}{2}+0 \\ \frac{WL^2}{12}+0 \\ 0 \\ 0 \end{Bmatrix}$$

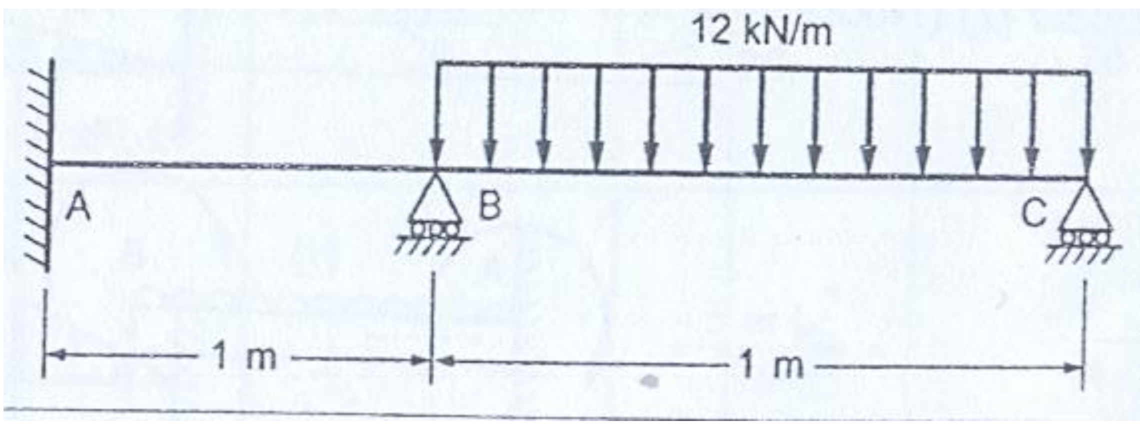
Neglecting the first, second, third, fifth and sixth row and column from the above equation,

$$\frac{EI}{L^3}\left[8L^2\right]\{\phi_2\} = \left\{\frac{WL^2}{12}\right\}$$

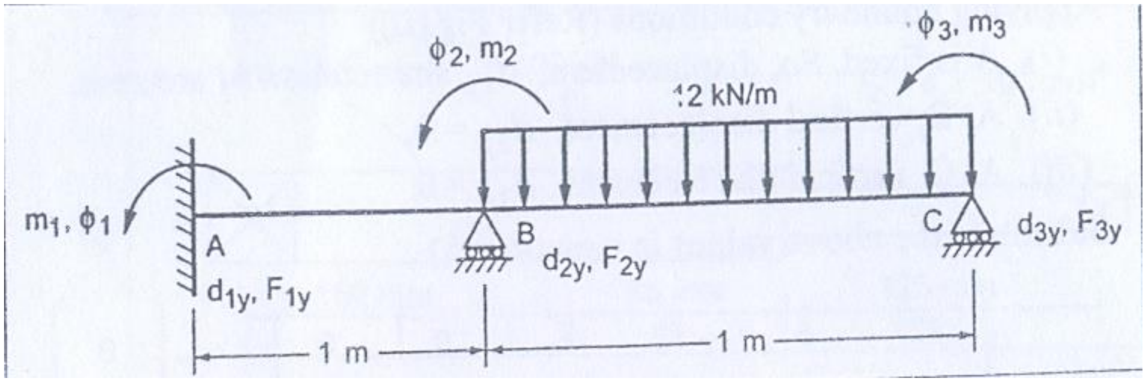
$$\phi_2 = \frac{WL^3}{96EI}$$

Slope or rotation, $\phi_2 = \dfrac{WL^3}{96EI}$

29. For the beam and loading down in fig. calculate the rotations B and C. E=210Gpa, I=6x10^{6}mm^4.



Given:



Young's modulus, E = 210 GPa = 210 * 10^9 N/m^2

Moment of inertia, I = 6 *10^6 mm^4 = 6 * 10^{-6} mm^4

Element 1, length L_1 = 1 m

` element 2, length L_2 = 1 m

Uniformly distribute load, W = 12 KN/m = 12*10^3 N/m

To find

i. Slope or rotation at B, ϕ_2
ii. Slope or rotation at C, ϕ_3

Solution:

We can divide the beam into two element

For element 1: node 1, 2, 3 and 4; d_{1y}, ϕ_1, d_{2y}, ϕ_2)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix}$$

There is no load and moment in element 1.

$F_{1y} = 0$, $m_1 = 0$, $F_{2y} = 0$, $m_2 = 0$

$$\frac{210*10^9*6*10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For element 2: node 3,4,5 and 6; d_{2y}, ϕ_2, d_{3y}, ϕ_3)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix}$$

For uniformly distributed load,

F_{2y} = -WL/2 = -12*10^3*1/2 = -6000 N

F_{3y} = -WL/2 = -12*10^3*1/2 = -6000 N

m_2 = -WL^2/12 = -12*10^3*1^2/12 = -1000 N-m

m_3 = WL^2/12 = 12*10^3*1^2/12 = 1000 N-m

$$\frac{210*10^9*6*10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix}$$

assemble the finite element

$$\frac{210*10^9*6*10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 12+12 & -6+6 & -12 & 6 \\ 6 & 2 & -6+6 & 4+4 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0-6000 \\ 0-1000 \\ -6000 \\ 1000 \end{Bmatrix}$$

$$1.26*10^6\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix}$$

Applying boundary condition;

- A is fixed, displacement d_{1y} and rotation ϕ_1 are 0
- At B, vertical displacement, d_{2y} = 0
- At C, vertical displacement, d_{3y} = 0

Substitute the boundary condition in the global stiffness matrix,

$$1.26*10^6\begin{bmatrix}12 & 6 & -12 & 6 & 0 & 0\\ 6 & 4 & -6 & 2 & 0 & 0\\ -12 & -6 & 24 & 0 & -12 & 6\\ 6 & 2 & 0 & 8 & -6 & 2\\ 0 & 0 & -12 & -6 & 12 & -6\\ 0 & 0 & 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\0\\0\\\phi_2\\0\\\phi_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\-6000\\-1000\\-6000\\1000\end{Bmatrix}$$

Neglecting the first, second, third and fifth row and column from the above equation,

$$1.26*10^6\begin{bmatrix}8 & 2\\ 2 & 4\end{bmatrix}\begin{Bmatrix}\phi_2\\\phi_3\end{Bmatrix}=\begin{Bmatrix}-1000\\1000\end{Bmatrix}$$

$1.26 * 10^6 [8\phi_2 + 2\phi_3] = -1000$

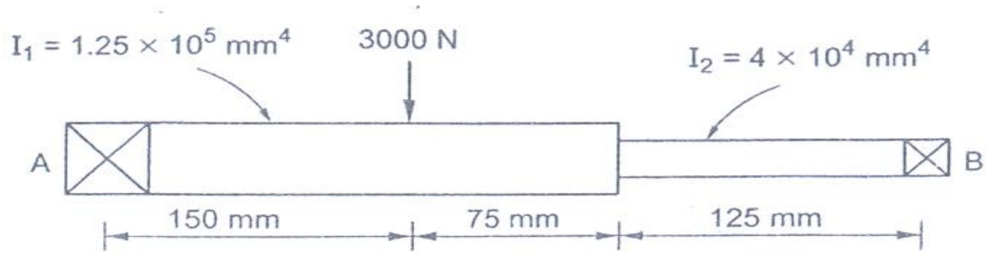
$1.26 * 10^6 [2\phi_2 + 4\phi_3] = 1000$

Solving the above equation

$\phi_2 = -1.70 * 10^{-4}$ rad

$\phi_3 = 2.834 * 10^{-4}$ rad

30. Find the deflection at the point load and slope at the ends for the steel shaft which simply supported at the bearing and B as shown in fig. take E=200Gpa.

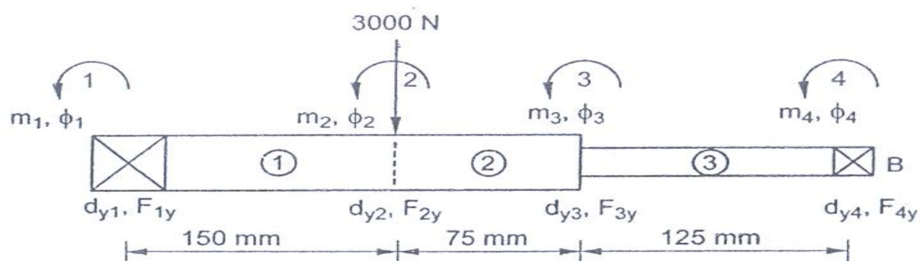


Given:

Young's modulus, E = 200 GPA = $2*10^5$ N/mm^2

Moment of inertia for element 1 & 2, $I_1 = 1.25*10^5$ mm^4

Element 3, $I_2 = 4*10^5$ mm^4



Element 1, length L_1 = 150 mm

Element 2, length L_2 = 75 mm

Element 3, length L_3 = 125 mm

Point load at node 2, F_2 = 3000N

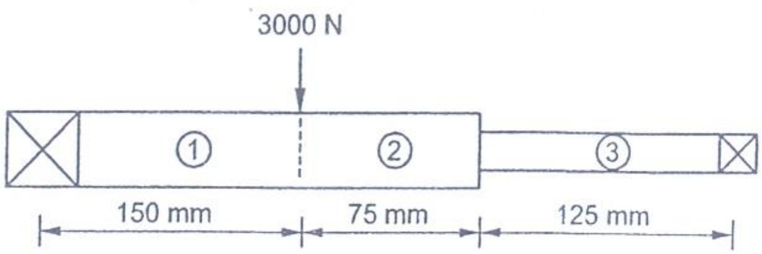
To find:

i. Deflection at the point load

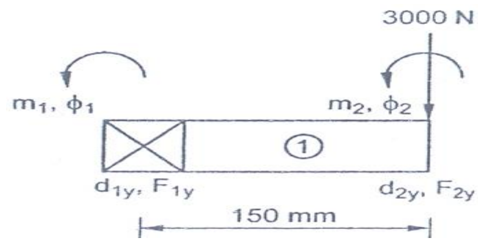
ii. Slope at the ends

Solution:

We can divide the beam into three elements.



For element 1: node 1, 2, 3 and 4; d_{1y}, ϕ_1, d_{2y}, ϕ_2)



Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix}$$

There is no load and moment in element 1.

$F_{1y} = F_{1y}$, $m_1 = 0$, $F_{2y} = -3000N$, $m_2 = 0$

$$\frac{2*10^5*1.25*10^5}{150^3}\begin{bmatrix} 12 & 900 & -12 & 900 \\ 900 & 9*10^4 & -900 & 4.5*10^4 \\ -12 & -900 & 12 & -900 \\ 900 & 4.5*10^4 & -900 & 9*10^4 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ 0 \\ -3000 \\ 0 \end{Bmatrix}$$

$$10^4\begin{bmatrix} 8.88 & 666 & -8.88 & 666 \\ 666 & 66600 & -666 & 33300 \\ -8.88 & -666 & 8.88 & -666 \\ 666 & 33300 & -666 & 66600 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -3000 \\ 0 \end{Bmatrix}$$

For element 2: node 3,4,5 and 6; d_{2y}, ϕ_2, d_{3y}, ϕ_3)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix}$$

There is no load and moment in element 1.

$F_{2y} = 0$, $m_2 = 0$, $F_{3y} = 0$, $m_3 = 0$

$$\frac{2*10^5*1.25*10^5}{75^3}\begin{bmatrix} 12 & 450 & -12 & 450 \\ 450 & 22500 & -450 & 11250 \\ -12 & -450 & 12 & -450 \\ 450 & 11250 & -450 & 22500 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix}$$

$$10^4\begin{bmatrix} 8.88 & 666 & -8.88 & 666 \\ 666 & 66600 & -666 & 33300 \\ -8.88 & -666 & 8.88 & -666 \\ 666 & 33300 & -666 & 66600 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For element 3: node 5, 6, 7 and 8; d_{3y}, ϕ_3, d_{4y}, ϕ_4)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{3y} \\ \phi_3 \\ d_{4y} \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} F_{3y} \\ m_3 \\ F_{4y} \\ m_4 \end{Bmatrix}$$

There is no load and moment in element 1.

$F_{3y} = 0$, $m_3 = 0$, $F_{4y} = F_{4y}$, $m_4 = 0$

$$\frac{2*10^5*4*10^5}{125^3}\begin{bmatrix} 12 & 450 & -12 & 450 \\ 450 & 22500 & -450 & 11250 \\ -12 & -450 & 12 & -450 \\ 450 & 11250 & -450 & 22500 \end{bmatrix}\begin{Bmatrix} d_{3y} \\ \phi_3 \\ d_4 y \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} F_{3y} \\ m_3 \\ F_{4y} \\ m_4 \end{Bmatrix}$$

$$10^4\begin{bmatrix} 4.92 & 307.5 & -4.92 & 307.5 \\ 307.5 & 25625 & -307.5 & 12812.5 \\ -4.92 & -307.5 & 4.92 & -307.5 \\ 307.5 & 12812.5 & -307.5 & 2565 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{4y} \\ 0 \end{Bmatrix}$$

Assembling the finite element equation

$$10^4\begin{bmatrix} 8.88 & 666 & -8.88 & 666 & 0 & 0 & 0 & 0 \\ 666 & 66600 & -666 & 33300 & 0 & 0 & 0 & 0 \\ -8.88 & -666 & 80.04 & 2002.5 & -71.16 & 2668.5 & 0 & 0 \\ 666 & 33300 & 2002.5 & 200025 & -2668.5 & 066712.5 & 0 & 0 \\ 0 & 0 & -71.16 & -2668.5 & 76.08 & -2361 & -4.92 & 307.5 \\ 0 & 0 & 2668.5 & 66712.5 & -2361 & 159050 & -307.5 & 12812.5 \\ 0 & 0 & 0 & 0 & -4.92 & -307.5 & 4.92 & -307.5 \\ 0 & 0 & 0 & 0 & 307.5 & 12812.5 & -307.5 & 25625 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ d_{4y} \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ 0 \\ -3000 \\ 0 \\ 0 \\ 0 \\ F_{4y} \\ 0 \end{Bmatrix}$$

Applying boundary conditions:

$d_{1y}= 0$, $\phi_1 = \phi_1$, $d_{2y} = d_{2y}$, $\phi_2 = \phi_2$

$d_{3y} = d_{3y}$, $\phi_3 = \phi_3$, $d_{4y} = 0$, $\phi_4 = \phi_4$

since $d_{1y} = 0$, $d_{4y} = 0$, neglect the first and seventh row and column in the global stiffness matrix.

$$10^4\begin{bmatrix} 66600 & -666 & 33300 & 0 & 0 & 0 \\ -666 & 80.04 & 2002.5 & -71.16 & 2668.5 & 0 \\ 33300 & 2002.5 & 200025 & -2668.5 & 66712.5 & 0 \\ 0 & -71.16 & -2668.5 & 76.08 & -2361 & 307.5 \\ 0 & 2668.5 & 66712.5 & -2361 & 159050 & 12812.5 \\ 0 & 0 & 0 & 307.5 & 12812.5 & 25625 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Convert the above matrix into equation,

$10^4[66600\ \phi_1 - 666\ d_{2y} + 33300\ \phi_2] = 0$

$10^4[-666\phi_1 + 80.04 d_{2y} + 2002.5\phi_2 - 71.16\ d_{3y} + 2668.5\ \phi_3] = -3000$

$10^4[\ 33300\phi_1 + 2002.5\ d_{2y} + 200025\ \phi_2 - 2668.5\ d_{3y} + 66712.5\ \phi_3] = 0$

$10^4[\ -71.16\ d_{2y} - 2668.5\ \phi_2 + 76.08\ d_{3y} - 2361\ \phi_3 + 307.5\ \phi_4] = 0$

$10^4[\ 2668.5\ d_{2y} + 66712.5\ \phi_2 - 2361\ d_{3y} + 159050\ \phi_3 + 12812.5\ \phi_4] = 0$

$10^4[\ 307.5\ d_{3y} + 12812.5\ \phi_3 + 25625\ \phi_4] = 0$

Solving the above equation,

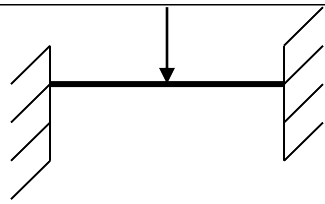
$\phi_1 = -0.0011$ rad

$\phi_4 = 0.0010$ rad

$d_{2y} = -0.112$ mm

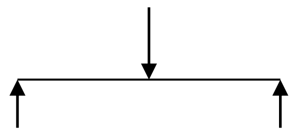
<u>TRANSVERSE DEFLECTION OF BEAM</u>

31. Explain About The Transverse Deflection Of Beam



Fixed-Fixed beam : Deflections and slope at both ends are zero which constitute geometric boundary conditions

Free-free beam Bending moment and shear forces are zero at both ends (natural boundary conditions at both ends)

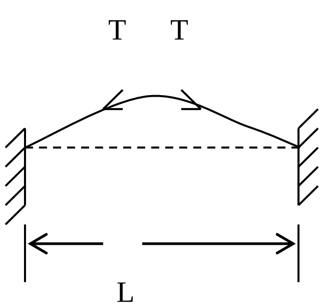


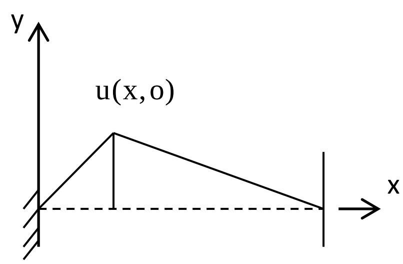
Simply supported beam

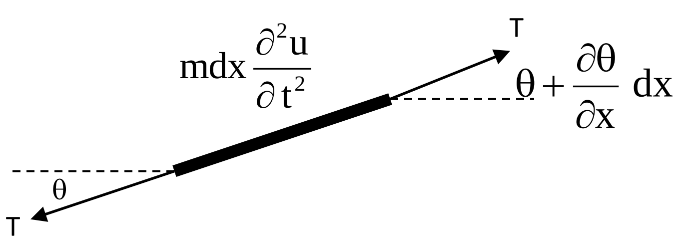
(mixed boundary conditions)

Natural boundary condition also known as additional or dynamic boundary condition, which results from the balance of moments or forces in the boundary. For example in case of a free-free beam which may be a model of a flying aeroplane or a spacecraft, at both the ends in this system shear force and bending moments are zero. Hence they constitute the natural boundary conditions.

Lateral vibration of a flexible taut string







Strings are mostly used in musical instruments and many other applications of domestic and industrial in nature. A string of length *L* is shown in Figure, which is subjected to tension *T*. Let at time t = 0, the string is pulled in the lateral direction (y direction) as shown in figure and left. Hence the lateral deflection *u* along the string is a function of the space variable *x* and time *t* i.e., $u = u(x,t)$

If the lateral deflection is small, the change in tension *T* due to the deflection is negligible. Figure 10.4(c) shows the free body diagram of an elemental length dx of the string. When the string is vibrating, in the y direction inertia force is acting. Considering the forces in the vertical direction, applying Newton's second law one may have

$$mdx\frac{\partial^2 u}{\partial t^2} = T\sin\left(\theta + \frac{\partial \theta}{\partial x}dx\right) - T\sin\theta$$

Here *m* is the mass per unit length of the string. Now assuring small deflections *u* and slope θ, the equation reduces to following equation.

$$mdx\frac{\partial^2 u}{\partial t^2} = T\left(\theta + \frac{\partial \theta}{\partial \mathrm{x}}.dx\right) - T\theta$$

Now substituting the slope $\theta = \frac{\partial u}{\partial x}$ in equation one may write

$$mdx\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}dx$$

or, $$m\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}$$

or, $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\frac{\partial^2 u}{\partial x^2}$$

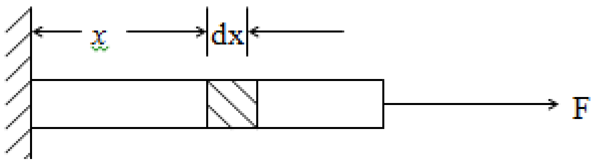
or, $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad where,\ c = \sqrt{\frac{T}{m}}$$

The above equation is known as Wave equation. One may use Hamilton's principle to derive the equation of motion of this system also.

32. Discuss about the Longitudinal Vibration.

LONGITUDINAL VIBRATION:

Consider a rod of length *l* (Fig.) subjected to a force F at time t=0 and then released. It will be subjected to longitudinal vibration.



Longitudinal vibration of rod

Let $u(x,t)$ be the axial displacement of an element dx of the rod.

From Hook's law $\frac{P}{A} = E\frac{\partial u}{\partial x}$ (1)

Applying Newton's second law

$$dm\frac{\partial^2 u}{\partial t^2} = \left(p + \frac{\partial p}{\partial x}dx\right) - p \quad (2)$$

$$\Rightarrow \rho \text{ Adx}\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)dx \quad (3)$$

P= force at x

A= Cross sectional area

E= Young's Modulus.

d $dm = \rho Adx$ is the mass of the element dx

ρ = Mass per unit volume.

From equation (3) <u>If AE is const</u>

$$\frac{\partial^2 u}{\partial t^2} = \frac{EA}{\rho A}\frac{\partial^2 u}{\partial x^2}$$

or $\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$

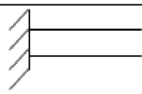
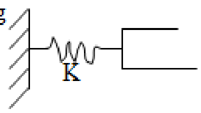
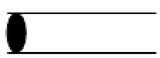
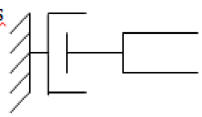
or $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ where $c = \sqrt{\dfrac{E}{\rho}}$

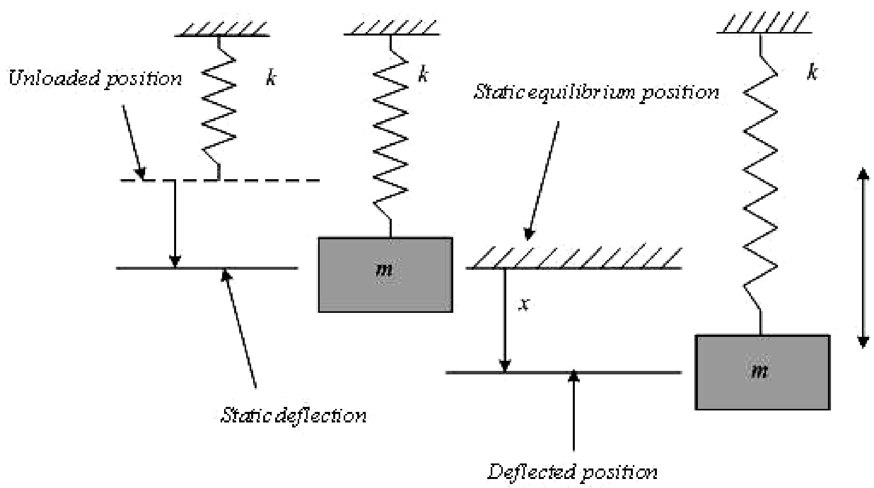
It may be observed that we are getting the same wave equation in this case where only c is different. It can be shown that c represent the velocity of the wave in the rod.

33. Discuss about the various mode shapes exerted by the beam while vibration.

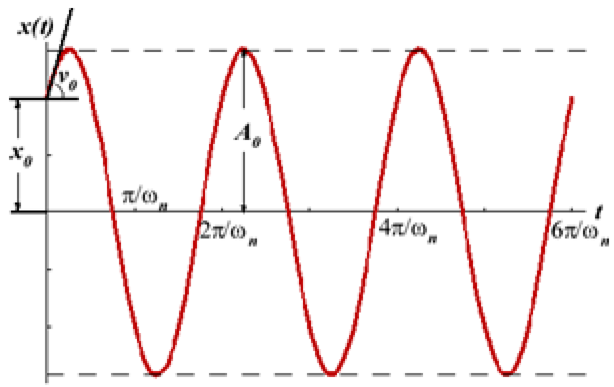
MODE SHAPES:

Boundary Conditions for the longitudinal vibration of rod

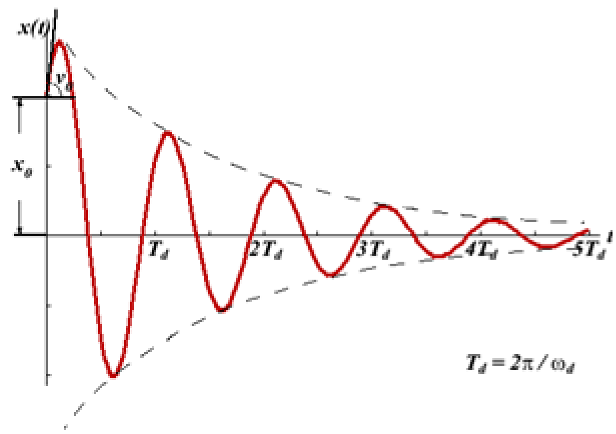
Case	Boundary condition left $x=0$	Boundary condition right $x=l$
Free end 	$\frac{\partial u}{\partial x} = 0$	$\frac{\partial u}{\partial x} = 0$
Fixed end 	$u(0,t) = 0$	$u(L,t) = 0$
End spring 	$AE\frac{\partial u}{\partial x} = Ku$	$AE\frac{\partial u}{\partial x} = -ku$
End man 	$AE\frac{\partial u}{\partial x} = m\frac{\partial^2 u}{\partial t^2}$	$AE\frac{\partial u}{\partial x} = -m\frac{\partial^2 u}{\partial t^2}$
End dampes 	$AE\frac{\partial u}{\partial x} = c\frac{\partial u}{\partial t}$	$AE\frac{\partial u}{\partial x} = -c\frac{\partial u}{\partial t}$

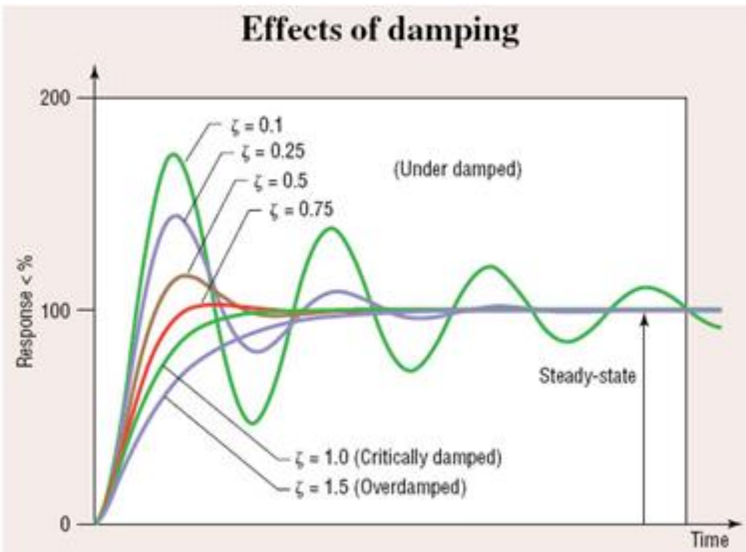


Free undamped vibration



Free damped vibration





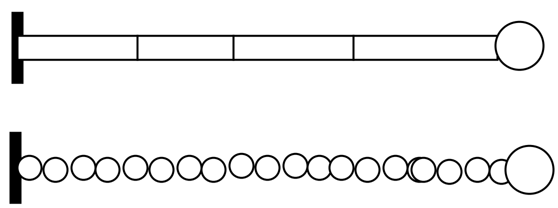
34. Write short notes on the natural frequency of beam.

NATURAL FREQUENCY:

The natural frequency can be calculated using the formula $\omega_n = \sqrt{\frac{k}{m}}$ where m is the attached mass. In this calculation we have neglected the mass of the beam.

Hence it may be observed that by considering a point mass at the tip we obtained one natural frequency of the system. Instead of modeling this system as a single-spring mass if one consider the beam to be consist of several masses, then the system can be modeled as a multi-degree of freedom system as shown in figure .

But as the dimension of each elemental mass considered in the above case is arbitrary, one may consider the beam as a continuous system with infinity number of distributed mass and stiffness and hence has infinity number of natural frequencies.



- So in contrast to the discrete mass system, in distributed mass or continuous system the system has infinite number of natural frequencies and corresponding to each natural frequency, the system will have a distinct mode shape.
- It may be observed that the response of the continuous system depends time and space coordinate (location). But in case of discrete system the response is only a function of time. Hence while the equation of motion of discrete systems are written in terms of ordinary differential equations, in case of continuous system they are written in terms of partial differential equations.
- It may be noted that all the real systems are continuous system.

[m]= ρAl/6$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Two Marks Question and Answers.

UNIT-II

1. **What are the methods are generally associated with the finite element analysis?**
- Force method
- Displacement or stiffness method.

2. **Explain stiffness method.**

Displacement or stiffness method, displacement of the nodes is considered as the unknown of the problem. Among them two approaches, displacement method is desirable.

3. **Explain Force Method?**

In force Method, internal forces are considered as the unknowns of the problem

4. **What is meant by discretization?**

The art of subdividing a structure in to convenient number of smaller components is known as discretization. These smaller components are then put together. The process of uniting the various elements together is called assemblage.

5. **What is meant by Assemblage?**

These smaller components are then put together. The process of uniting the various elements together is called assemblage.

6. **What is truss element? (Nov2012)**

The truss elements are the part of a truss structure linked together by point joint which transmits only axial force to the element.

7. **During discretization, mention the places where it is necessary to place a node?**
- Concentrated load acting point
- Cross-section changing point
- Different material interjections point
- Sudden change in point load

8. **What is the difference between static and dynamic analysis?**

Static analysis: The solution of the problem does not vary with time is known as static analysis
Example: stress analysis on a beam

Dynamic analysis: The solution of the problem varies with time is known as dynamic analysis
Example: vibration analysis problem.

9. **What are the types of loading acting on the structure?**
- Body force (f)
- Traction force (T)
- Point load (P)

10. **Define the body force**

A body force is distributed force acting on every elemental volume of the body
Unit: Force per unit volume.
Example: Self weight due to gravity

11. **Define traction force**

Traction force is defined as distributed force acting on the surface of the body.

Unit: Force per unit area.
Example: Frictional resistance, viscous drag, surface shear

12. What is point load?
Point load is force acting at a particular point which causes displacement.

13. Write down the general finite element equation.
□□□□□□□□□□□□□□
K-Stiffness matrix in N/mm
U-Nodal displacement in mm

14. What do you mean by constitutive law?
For a finite Element, the stress-strain relations are expressed as follows:

$\{\sigma\}= \{D\}\{e\}$

$\{\sigma\}$= Stress in N/m^2
{D}=Stress-Strain relationship matrix
{e}=Strain (No Unit)

15. What is interpolation functions? (May 2012)

The function used to represent the behavior of the field variable with in an element are called interpolation functions.

16. What are the methods are generally associated with the finite element analysis?

17. Force method
18. Displacement or stiffness method.

19. Explain stiffness method.
Displacement or stiffness method, displacement of the nodes is considered as the unknown of the problem. Among them two approaches, displacement method is desirable.

20. Explain Force Method?
In force Method, internal forces are considered as the unknowns of the problem

21. What are the classifications of coordinates?
Global coordinates

Local coordinates

Natural coordinates

22. What is Global coordinates?
The points in the entire structure are defined using coordinates system is known as global coordinate system.

23. What is natural coordinates?
A natural coordinate system is used to define any point inside the element by a set of dimensionless number whose magnitude never exceeds unity. This system is very useful in assembling of stiffness matrices.

24. Define shape function.

Approximate relation φ (x,y) = N1 (x,y) φ1 + N2 (x,y) φ2 + N3 (x,y) φ3

Where φ1, φ2, and φ3 are the values of the field variable at the nodes N1, N2, and N3 are the interpolation functions.

N1, N2, and N3 are also called shape functions because they are used to express the geometry or shape of the element.

25. Distinguish between 1D bar element and 1D beam element (Nov 2009)

- 1-D bar element bar element has axial deformation {u} and the element stiffness matrix is 2*2
- 1-D beam Element has transverse deformation and rotation and the element stiffness matrix is 4*4

26. **What are the characteristic of shape function?**

It has unit value at one nodal point and zero value at other nodal points. The sum of shape function is equal to one.

27. **Why polynomials are generally used as shape function?**

Differentiation and integration of polynomial are quite easy.

The accuracy of the result can be improved by increasing the order of the polynomial. It is easy to formulate and computerize the finite element equations.

28. **How do you calculate the size of the global stiffness matrix?**

Global stiffness matrix size=Number of nodes X Degrees of freedom per node

29. Write down the expression of stiffness matrix for one dimensional bar element

[k]=AE (1 -1)

l (-1 1)

30. **State the properties of stiffness matrix**

- It is asymmetric matrix
- The sum of elements in any column must be equal to zero.
- It is an unstable element. So the determinant is equal to zero.

31. **Write down the expression of shape function N and displacement u for one dimensional bar element.**

$U=N_1u_1+N_2u_2$

$N_1=1-X/l$

$N_2 = X/l$

32. **Define total potential energy.**
Total potential energy, π=Strain energy (U)+ potential energy of the external forces (W)

33. **State the principle of minimum potential energy.**
Among all the displacement equations that satisfied internal compatibility and the boundary condition those that also satisfy the equation of equilibrium make the potential energy a minimum is a stable system.

34. **Write down the finite element equation for one dimensional two noded bar element.**

[F]= [K] {u}

F- Force in N

K- Stiffness Matrix in N/mm

u- Displacement in mm

35. **What is truss?**
A truss is defined as a structure made up of several bars, riveted or welded together.

36. **Statetheassumptionaremadewhilefindingtheforcesinatruss**
Allthemembersarepinjointed. Thetrussisloadedonlyatthejoint

Theself-weightofthemembersisneglectedunlessstated.

37. **Statetheprinciplesofvirtualenergy?**
Abodyisinequilibriumiftheinternalvirtualworkequalstheexternalvirtualwork fortheeverykinematicallyadmissibledisplacementfield.

38. **Whatisessentialboundarycondition**
Primary boundary condition orEBC Boundary condition which interms offield variableisknownasPrimaryboundarycondition.

39. **State Naturalboundaryconditions**
Secondaryboundarynaturalboundaryconditionswhichareinthe differentialform offieldvariableisknownassecondaryboundarycondition

40. **State the assumptions made while finding the forces in a truss (Nov 2011)**
    - All the members are pin jointed
    - The truss is loaded only at the joints
    - The self-weights of the members are neglected unless stated

41. **Whendo we resort to 1D quadratic spar (bar) elements? (May 2011)**

- Better Accuracy
- Representation of curved boundaries
- Faster Convergence

42. **Whatismeantbydegreesoffreedom?**

Whentheforceor reactionactatnodalpointnodeis subjectedtodeformation.The deformation includesdisplacement rotation, andorstrains. These arecollectively knownasdegreesoffreedom.

43. **Statetheprinciplesofvirtualenergy?**

Abodyisinequilibriumiftheinternalvirtualworkequalstheexternalvirtualwork fortheeverykinematicallyadmissibledisplacementfield.

44. **Whatishomogeneousform?**

Whenthespecifiedvaluesofdependentvariablesiszero,theboundaryconditionare saidtobehomogeneous

45. **Whatisnon-homogeneousform?**

Whenthespecifiedvaluesofdependentvariablesarenon-zero,theboundaryconditisaidtobenon-homogeneous.

46. **Define frequency of vibration.**

It is the number of cycles described in one second. Unit is HZ

47. **Define damping ratio.**

It is define as the ratio of actual damping coefficient (c) to the critical damping coefficient (c_c)

$$\text{Damping ratio } \varepsilon = \frac{c}{c_c} = \frac{c}{2m\omega_n}0$$

48. **What is meant by longitudinal vibration?**

When the particles of the shaft or disc moves parallel to the axis of the shaft, then the vibrations are known as longitudinal vibration.

49. **What is meant by transverse vibration?**

When the particles of the shaft or disc moves approximately perpendicular to the axis of the shaft, then the vibrations are known as transverse vibration.

50. **Define magnification factor.**

The ratio of the maximum displacement of the forced vibration (x_{max}) to the static deflection under the static force (x_0) is known as magnification factor.

51. **Write down the expression of longitudinal vibration of bar element.**

Free vibration equation for axial vibration of bar element is

$$[K]\{u\} = \omega^2[m]\{u\}$$

Where, u - displacement

$[K]$– stiffness matrix

$$[K] - \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

ω – natural frequency

$[m]$ – mass matrix

$$\text{Lamped } [m] = \frac{\rho Al}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consistent $[m] = \frac{\rho AL}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

52. Write down the expression of governing equation for free axial vibration of rod.

The governing equation for free axial vibration of rod is given by,

$$AE\,\frac{\partial^2 u}{\partial x^2} = \rho A\,\frac{\partial^2 u}{\partial t^2}$$

Where, E – Young's modulus,

A – cross section area

ρ – density

53. Write down the expression of governing equation for transverse vibration of beam

The governing equation for free transverse vibration of a beam is

$$EI\,\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0$$

Where, E –Young's modulus

I –moment of inertia

ρ –density

A –cross sectional area

54. **Write down the expression of transverse vibration of beam element.**

Free vibration equation for transverse vibration of beam element is,

$$[K]\{u\} = \omega^2[m]\{u\}$$

Where, $[K]$ = stiffness matrix for beam element

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$[m]$ = mass matrix

$$[m] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} for\ consistent\ mass\ matrix$$

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} for\ lumped\ mass\ matrix$$

St.Anne's College of Engineering & Technology,
Department of Mechanical Engineering

Subject Name	:	**FINITE ELEMENT ANALYSIS**
Subject code	:	**ME8692**
Year	:	**IIIrd year**
Semester	:	**VIth semester**

UNIT III

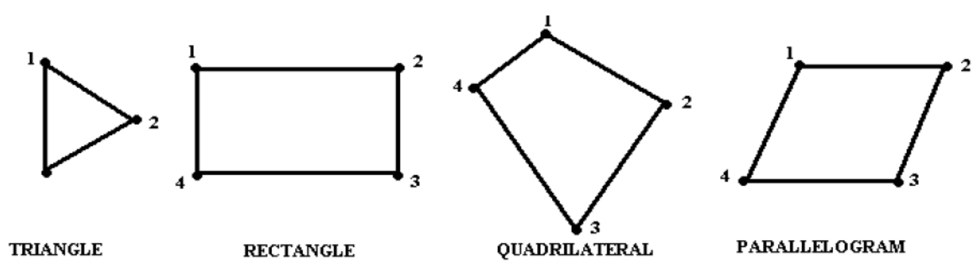
TWO DIMENSIONAL SCALAR VARIABLE PROBLEMS

Second Order 2D Equations involving Scalar Variable Functions – Variational formulation –Finite Element formulation – Triangular elements – Shape functions and element matrices and vectors.Application to Field Problems - Thermal problems – Torsion of Non circular shafts – Quadrilateral elements – Higher Order Elements.

1. **Write short notes on second order 2D equations involving scalar variable functions**

SECOND ORDER 2D EQUATIONS INVOLVING SCALAR VARIABLE FUNCTIONS

Two dimensional elements Two dimensional elements are defined by three or more nodes in a two dimensional plane (i.e., x, y plane). The basic element useful for two dimensional analysis is the triangular element.



Many engineering structures and mechanical components are subjected to loading in two directions. Shafts, gears, couplings, mechanical joints, plates, bearings, are few examples. Analysis of many three dimensional systems reduces to two dimensional, based on whether the loading is plane stress or plane strain type.

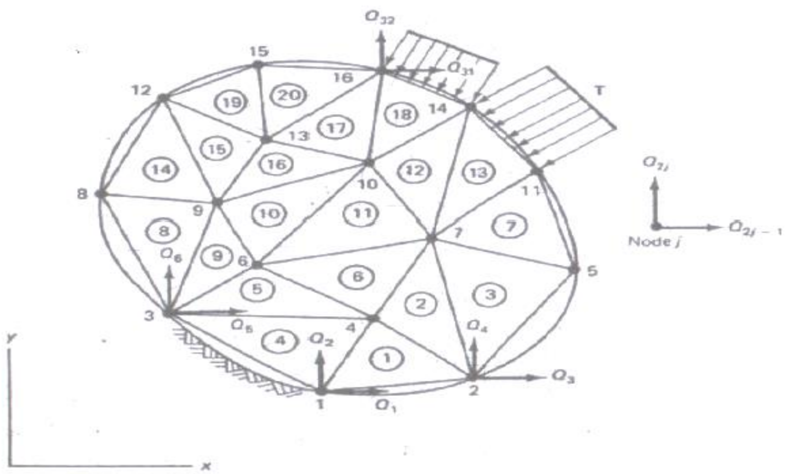
Triangular elements or Quadrilateral elements are used in the analysis of such components and systems. The various load vectors, displacement vectors, stress vectors and strain vectors used in the analysis are as written below, the displacement vector $u = [u, v]^T$,

u is the displacement along x direction, v is the displacement along y direction,the body force vector $f = [fx , fy]^T$ fx , is the component of body force along x direction, fy is the component of body force along y direction. The traction force vector T = [T_x , T_y]^T Tx , is the component of body force along x direction, Ty is the component of body force along y direction.

The element having mid side nodes along with corner nodes is a higher order element.

Element having a curved side is also a higher order element. A simple quadrilateral element has straight edges and corner nodes. This is also a linear element. It can have constant thickness or variable thickness. The quadrilateral having mid side nodes along with corner nodes is a higher order element. Element having a curved side is also a higher order element.

The given two dimensional componentsis divided in to number of triangular elements or quadrilateral elements. If the component has curved boundaries certain small region at the boundary is left uncovered by the elements. This leads to some error in the solution.



Two dimensional stress strain equations
From theory of elasticity for a two dimensional body subjected to general loading the equations of equilibrium are given by

$$[\partial\sigma_x/\partial x] + [\partial\tau_{yx}/\partial y] + F_x = 0$$

$$[\partial\tau_{xy}/\partial x] + [\partial\sigma_y/\partial y] + F_y = 0$$

Also $\tau_{xy} = \tau_{yx}$

The strain displacement relations are given by

$$\varepsilon_x = \partial u/\partial x, \quad \varepsilon_y = \partial v/\partial y, \quad \gamma_{xy} = \partial u/\partial y + \partial v/\partial x$$

$$\varepsilon = [\ \partial u/\partial x,\ \partial v/\partial y,\ (\partial u/\partial y + \partial v/\partial x)\ ]^T$$

The stress strain relationship for plane stress and plane strain conditions are given by thematrices shown in the next page. σ_x σ_y τ_{xy} ε_x ε_y γ_{xy} are usual stress straincomponents, v is the poisons ratio. E is young's modulus. Please note the differences in[D] matrix .

Plane Stress and Plane Strain

The 2D element is extremely important for the Plane Stress analysis and Plane Strain analysis.

Plane Stress Analysis:

It is defined to be a state of stress in which the normal stress (fã) and shear stress (fä) directed perpendicular to the plane are assumed to be zero.

Plane Strain Analysis:

It is defined to be a state of strain in which the normal to the xy plane and the shear strain are assumed to be zero.

VARIATIONAL FORMULATION

Variational formulation refers to the construction of a functional or a variational principle that is equivalent to the governing equations of the problem. It is nothing but the formation I which the governing equations are translated into equivalent weighted integral statements that are not necessarily equivalent to a variational principle.

It is common, especially in structural mechanics to express finite element formulation in vector notation (i.e., in terms of matrices).

We know that,

B(w,u) = l(w)

Let,

$$C = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ 1 \end{bmatrix}$$

B and l are expressed as,

$$\mathbf{B\ (w,u)} = \int \begin{bmatrix} \partial w/\partial x \\ \partial w/\partial y \\ w \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ u \end{bmatrix} \text{dx dy}$$

It is,B (w,u) = $\int (DW)^{T}$CD u dx dy

l(w) = $\int (W)^{T}$f dx dy + $\int W^{T}$qdsis the variational formulation of a two dimensional problem.

2. Explain the finite element formulation for 2D in detail

FINITE ELEMENT FORMULATION

The major steps are as follows

1. Discretization of domain into finite elements.
2. Weak formulation of governing differential equation.
3. Derivation of finite element interpolation functions.
4. Develop finite element model using weak form.
5. Assembly of finite elements to obtain the global system of algebraic equations.

6. Imposition of boundary conditions.
7. Solution of equations.
8. Post computation of solution.
   Here, steps 6 and 7 remain unchanged from one-dimensional finite element analysis.

The basic element useful for two dimensional analysis is the triangular element. The simple two dimensional elements have corner nodes.

a) Triangular element
b) Rectangular element
c) Quadratic element
d) Parallelogram element

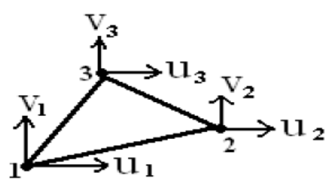
Triangular elements: Nodal displacements of the element represent the displacements at points inside an element. As discussed earlier, the finite element method uses the concept of shape function in systematically developing these interpolations.

Generally triangular elements are classified into two types, they are:

(i) Constant Strain Triangular element
(ii) Linear strain triangular element

Constant Strain Triangular (CST) Element

A three noded triangular element is known as constant strain triangular (CST) element. It has six unknown displacement degrees of freedom (u1,v1, u2,v2, u3,v3).



Shape function for the CST element

Shape function $N_1 = (p_1 + q_1x + r_1y) / 2A$

Shape function $N_2 = (p_2 + q_2x + r_2y) / 2A$

Shape function $N_3 = (p_3 + q_3x + r_3y) / 2A$

Displacement function for the CST element

Displacement function u = $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N1 & 0 & N2 & 0 & N3 & 0 \\ 0 & N1 & 0 & N2 & 0 & N3 \end{bmatrix} X \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \end{Bmatrix}$

Linear Strain Triangular (LST) Element:

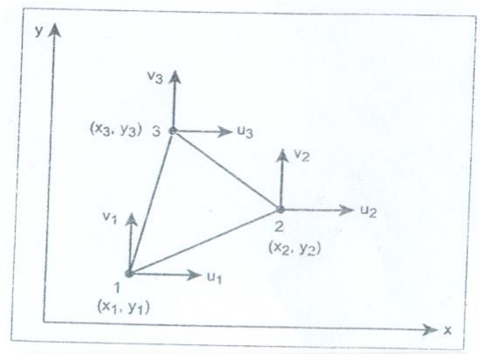
Linear strain triangular element is those, which has six nodes. The degrees of freedom of an LST element are twelve, because it has twelve unknown displacement.

The development of stiffness matrix in LST element is same as that of the CST element. The difference is that more number of equations and it is a tedious process to solve those equations. Hence it is solved by computer by using mathematical equations.

For plane stress applications, LST elements are preferred that the CST element. When large numbers of nodes are used, LST element is not preferred, since cost of formation of the element stiffness equation bandwidth is high compared to CST element.

3. **Derive the section with the development of the shape function for CST element for strain displacement matrix.(AU-NOV/DEC-2010)**

Consider a typical CST element with nodes 1,2 and3 as shown in fig. let the nodal displacement,



$$displacement, \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Since CST element has got two degrees of freedom at each node (u,v) the degree of freedom is 6. Hence it has 6 generalized coordinates.

Let

$$u = a_1 + a_2 x + a_3 y$$
$$v = a_4 + a_5 x + a_6 y$$

Where a1, a2, a3, a4, a5 and a6 are globalized coordinates

$$u_1 = a_1 + a_2 x_1 + a_3 y_1$$
$$u_2 = a_1 + a_2 x_2 + a_3 y_2$$
$$u_3 = a_1 + a_2 x_3 + a_3 y_3$$

Write above equation in matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$D = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

Let $D^{-1} = \dfrac{C^T}{|D|}$

The co-factors of matrix D is

$$C = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{vmatrix} \quad C^T = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix}$$

We know that

$$D = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$
$$|D| = 1(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)$$

Substitute C^T and D values in equation

$$D^{-1} = \frac{1}{1(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

substitute D^{-1} value in the above equation

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The area of the triangle can be expressed as function of the x, y coordinates of the nodes 1, 2 and 3

$$A = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$|A| = \frac{1}{2}[1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)]$$

$$2A = (x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)$$

Substitute 2A value in equation

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$p_1 = (x_2y_3 - x_3y_2)$$
$$p_2 = (x_3y_1 - x_1y_3)$$
$$p_3 = (x_1y_2 - x_2y_1)$$
$$q_1 = (y_2 - y_3)$$
$$q_2 = (y_3 - y_1)$$
$$q_3 = (y_1 - y_2)$$
$$r_1 = (x_3 - x_2)$$
$$r_2 = (x_1 - x_3)$$
$$r_3 = (x_2 - x_1)$$

From the equation we know that

$$u = a_1 + a_2x + a_3y$$

We can write this equation in matrix form

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$ value in the above equation

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \frac{1}{2A} \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \frac{1}{2A} \begin{bmatrix} p_1 + q_1x + r_1y & p_2 + q_2x + r_2y & p_3 + q_3x + r_3y \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} \frac{p_1 + q_1x + r_1y}{2A} & \frac{p_2 + q_2x + r_2y}{2A} & \frac{p_3 + q_3x + r_3y}{2A} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The equation in the form of

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Similarly

$$v = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Shape function

$$N_1 = \frac{p_1 + q_1x + r_1y}{2A}$$

$$N_2 = \frac{p_2 + q_2x + r_2y}{2A}$$

$$N_3 = \frac{p_3 + q_3x + r_3y}{2A}$$

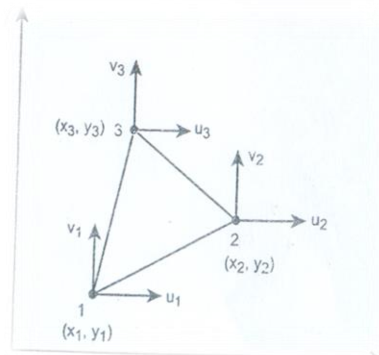
Assembling the equation in matrix form

Displacement function

$$u=\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

4. **Derive the Strain – Displacement matrix [B] for CST element.**

Strain – Displacement matrix [B] for CST element



Displacement function

$$u=\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$u=N_1u_1=N_2u_2+N_3u_3$

$v=N_1v_1=N_2v_2+N_3v_3$

The strain components for CST element are;

$$\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}$$

Where,

e_x and e_y --- Normal strain

γ_{xy} --- Shear Strain

Normal Strain, $e_x = \partial u / \partial x$

$$\partial u / \partial x = \partial / \partial x(\mathrm{N1u1} + \mathrm{N2u2} + \mathrm{N3u3})$$

Normal Strain, $e_y = \partial v / \partial y$

$$\partial v / \partial y = \partial / \partial y(\mathrm{N1v1} + \mathrm{N2v2} + \mathrm{N3v3})$$

Shear strain, $\gamma_{xy} = \partial u / \partial y + \partial v / \partial x$

$$\gamma_{xy} = \left(u_1(\partial \mathrm{N}_1/\partial y) + u_2(\partial \mathrm{N}_2/\partial y) + u_3(\partial \mathrm{N}_3/\partial y)\right) + \left(v_1(\partial \mathrm{N}_1/\partial x) + v_2(\partial \mathrm{N}_2/\partial x) + v_3(\partial \mathrm{N}_3/\partial x)\right)$$

On rearranging

$$\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 \\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y \\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

The above equation is of the form {e} = [B] (u)

We know that,

$$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

On substituting the shape functions we get,

$$\text{Strain – Displacement matrix [B]} = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

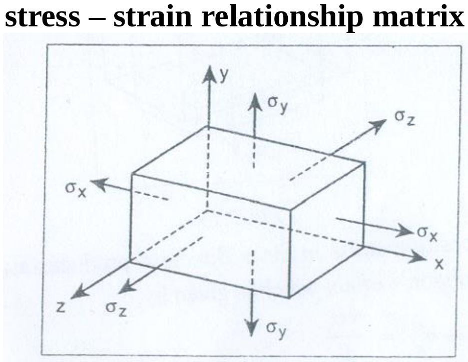
Where, $q_1 = y_2 - y_3$ $\quad r_1 = x_3 - x_2$

$q_2 = y_3 - y_1$ $\quad r_2 = x_1 - x_3$

$q_3 = y_1 - y_2$ $\quad r_3 = x_2 - x_1$

[B] is the strain displacement matrix for the CST element.This equation is the element strain equation.

5. Derive the stress – strain relationship matrix for two dimensional elements.



Hooke’s law states that when a material is loaded within its elastic limit, the stress is directly proportional to the strain.

Stress is directly proportional to strain

$\sigma \propto e$

σ=Ee

e=$\frac{\sigma}{E}$

where

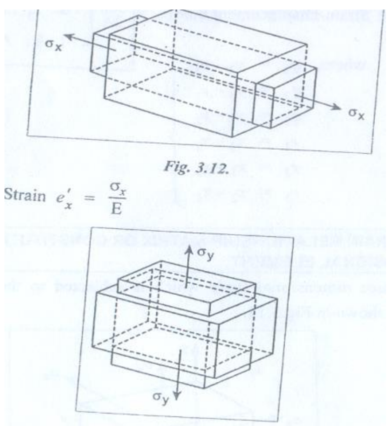
$e - strain$

$\sigma - stress$

$E - young's$ modulus

Strain $e'_x = \frac{\sigma_x}{E}$

The above fig shows the stress in y direction produces a negative strain in the x direction as a result of poisson's effect which is given by



-e"$_x = \frac{v\sigma_x}{E}$

e"$_x$ =-$\frac{v\sigma_x}{E}$

$v - poisson's$ ratio

Similarly the stress in the z direction produces a negative strain in the x direction as shown in fig

-e'''$_x = \frac{v\sigma_z}{E}$

e'''$_x = -\frac{v\sigma_z}{E}$

By applying superposition principle to the equation we get

$$e_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

The above equation is strain equation in x direction

Similarly the strain in y and z direction are

$$e_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$e_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Solving the above equation for normal stresses (σ_x, σ_y and σ_z).

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[e_x(1-v) + ve_y + ve_z\right]$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)}\left[ve_x + (1-v)e_y + ve_z\right]$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[ve_x + ve_y + (1-v)e_z\right]$$

The shear stress and shear strain relationship is given by,

$$\tau = G\gamma$$

The expressions for the three different sets of shear stresses are

$$\tau_{xy} = G\gamma_{xy}$$
$$\tau_{yz} = G\gamma_{yz}$$
$$\tau_{zx} = G\gamma_{zx}$$

where

$$G - \text{mod}\,ulus \text{ of rigidity} = \frac{E}{2(1+v)}$$

$$\tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1+v)(1-2v)}\left(\frac{1-2v}{2}\right)\gamma_{xy}$$

$$\tau_{yz} = \frac{E}{2(1+v)}\gamma_{yz}$$

$$\tau_{yz} = \frac{E}{(1+v)(1-2v)}\left(\frac{1-2v}{2}\right)\gamma_{yz}$$

$$\tau_{zx} = \frac{E}{2(1+v)}\gamma_{zx}$$

$$\tau_{zx} = \frac{E}{(1+v)(1-2v)}\left(\frac{1-2v}{2}\right)\gamma_{zx}$$

Assembling the above equation

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\frac{E}{(1+v)(1-2v)}\begin{bmatrix}1-v & v & v & 0 & 0 & 0\\ v & 1-v & v & 0 & 0 & 0\\ v & v & 1-v & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2}\end{bmatrix}\begin{Bmatrix}e_x\\ e_y\\ e_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}$$

The above equation is in the form of $\{\sigma\}=[D]\{e\}$

6. **Write down the Derivation for plane stress and plane strain in two dimensional planes.(AU-NOV/DEC-2011)**

PLANE STRESS

$$e_x=\frac{\sigma_x}{E}-v\frac{\sigma_y}{E}-v\frac{\sigma_z}{E}$$

$$e_y=-v\frac{\sigma_x}{E}+\frac{\sigma_y}{E}-v\frac{\sigma_z}{E}$$

For two dimensional plane stress problems, the normal stress, σ_z and the shear stress τ_{xz}, τ_{yz} are zero.

$\tau_{xz}=\tau_{yz}=\sigma_z=0$

The shear strain γ_{xz}, γ_{yz} are zero but $e_z\neq 0$

$\gamma_{xz}=\gamma_{yz}=0$

substitute $\sigma_z=0$ in e_x equation

$$e_x=\frac{\sigma_x}{E}-v\frac{\sigma_y}{E}\text{ -----------1}$$

substitute $\sigma_z=0$ in e_y equation

$$e_y=-v\frac{\sigma_x}{E}+\frac{\sigma_y}{E}\text{ ------------2}$$

Solving equation 1and 2

$$\sigma_x=\frac{E}{(1-v^2)}(e_x+ve_y)$$

$$\sigma_y=\frac{E}{(1-v^2)}(ve_x+e_y)$$

We know that,

Shear stress, $\tau_{xy} = G\ \gamma_{xy}$
G-modulus of rigidity = E/2(1+v)
γ_{xy}-shear strain
v-poisson's ratio

$$\tau_{xy} = \frac{E}{(1-v)(1+v)} \frac{(1-v)}{2} \gamma_{xy}$$

$$\tau_{xy} = \frac{E}{2(1+v)} \gamma_{xy} \quad \tau_{xy} = \frac{E}{(1-v^2)} \frac{(1-v)}{2} \gamma_{xy}$$

Arranging the equations in matrix form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

The above equation is in the form of{σ}=[D]{e}

The equation gives the two dimensional stress strain relationship for **plane stress problem**.

Where [D]=stress-strain relationship matrix

$$[D] = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

PLANE STRAIN

For plane strain we assume the following strain to be zero$e_z=\gamma_{xz}=\gamma_{yz}=0$

the shear stresses τxz=τyz=0, but $\sigma_z \neq 0$

from equation we know that

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Simplifying the above matrix we get

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

The above equation is in the form of {σ}=[D]{e}.

The equation gives the two dimensional stress strain relationship for **plane strain problem**

7. **Derive the stiffness matrix equation for two dimensional element (CST element)**
   We know that,

   Stiffness matrix, [K]= $\int_v [B][D][B]dv$

   $[k] = [B]^T [D][B]dv$

   Stiffness matrix, $[k] = [B]^T [D][B]dv$

   Where, A- area of the triangular element= $A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$

   t-thickness of the element

   [B]-strain-displacement matrix

   [B]= $\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

Where,

$$\begin{aligned} q_1 &= (y_2 - y_3) = 0 - 4 = -4 \\ q_2 &= (y_3 - y_1) = 4 - 0 = 4 \\ q_3 &= (y_1 - y_2) = 0 - 0 = 0 \\ r_1 &= (x_3 - x_2) = 1.5 - 3 = -1.5 \\ r_2 &= (x_1 - x_3) = 0 - 1.5 = -1.5 \\ r_3 &= (x_2 - x_1) = 3 - 0 = 3 \end{aligned}$$

[D]-stress-strain relationship matrix.

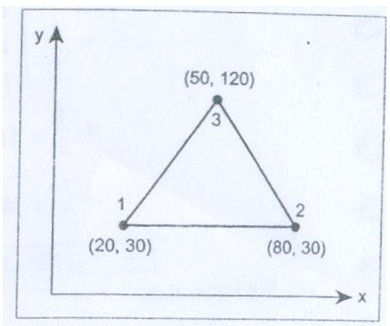
For plane stress problems,

$$[D]=\frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}$$

Where, E-young's or modulus of elasticity,

v-Poisson's ratio.

8. Determine the stiffness matrix for the CST element shown in fig. the coordinates are given in units of millimeters. Assume plane stress conditions. Take E=210 GPa, v = 0.25 and t = 10 mm. (April/May 2012)



Given

x_1=20mm; y_1=30mm

x_2=80mm; y_2=30mm

x_3=50mm; y_3=120mm

young's modulus, $E = 210\text{ GPa} = 210*10^9\text{Pa} = 210*10^3\text{N/mm}^2$

poisson's ratio, v = 0.25

thickness t = 10mm

To find: Stiffness matrix [k]

Solution: we know that stiffness matrix, $[k] = [B]^T [D] [B] At$

area $A=\frac{1}{2}\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}$ $A=\frac{1}{2}\begin{vmatrix}1 & 20 & 30\\ 1 & 80 & 30\\ 1 & 50 & 120\end{vmatrix}$

$$A=\frac{1}{2}\left[1(80*120-50*30)-20(120-30)+30(50-80)\right]$$
$$A=2700mm^2$$

We know that strain-displacement

$$[B]=\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$$q_1=(y_2-y_3)=30-120=-90$$
$$q_2=(y_3-y_1)=120-30=90$$
$$q_3=(y_1-y_2)=30-30=0$$
$$r_1=(x_3-x_2)=50-80=-30$$
$$r_2=(x_1-x_3)=20-50=-30$$
$$r_3=(x_2-x_1)=80-20=60$$

Substitute the above values in the matrix equation

$$[B]=\frac{1}{2A}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[B]=\frac{1}{2*2700}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[B]=\frac{30}{2*2700}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B]=5.55*10^{3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

We know that, Stress – strain relationship matrix for plane stress is

$$\left[D\right]=\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \qquad \left[D\right]=\frac{2.1*10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = 56*10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D][B] = 56*10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} *5.55*10^3 \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 311.08 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 311.08 \begin{bmatrix} -12 & 1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

We know that

$$[B] = 5.55*10^3 \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix} [B]^T = 5.55*10^3 \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$[B]^T [D][B] = 5.55*10^3 \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} 311.08 \begin{bmatrix} -12 & 1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

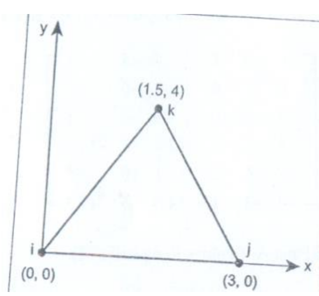
$$[B]^T\left[D\right][B] = 1.728\begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

Substitute $[B]^T$ [D] [B] and A, t values in equation

$$[k] = 1.728\begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix} * 2700 * 10 N/mm$$

$$[k] = 46.656 * 10^3\begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix} N/mm$$

9. **Evaluate the stiffness matrix for the CST element shown in fig. the coordinates are given in units of millimeters. Assume plane stress conditions. Take E=2*10^5 N/mm^2, v = 3 and t = 10 mm. (AU-Nov/Dec-2013, Jan 2006)**



Given

x_1=0mm; y_1=0mm

x_2=3mm; y_2=0mm

x_3=1.5mm; y_3=4mm

young's modulus, E = 2*10^{5}N/mm^2, poisson's ratio, v = 0.3, thickness t = 10mm

To find: stiffness matrix [k]

Solution: we know thatstiffness matrix, [k] = $[B]^T$ [D] [B] At

$$A=\frac{1}{2}\begin{vmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 1 & 1.5 & 4\end{vmatrix} \quad A=\frac{1}{2}\left[1(12-0)-0+0\right] \quad A=6mm^2$$

We know that strain-displacement

$$[B]=\frac{1}{2A}\begin{bmatrix}q_1 & 0 & q_2 & 0 & q_3 & 0\\ 0 & r_1 & 0 & r_2 & 0 & r_3\\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3\end{bmatrix}$$

$$q_1=(y_2-y_3)=0-4=-4$$
$$q_2=(y_3-y_1)=4-0=4$$
$$q_3=(y_1-y_2)=0-0=0$$
$$r_1=(x_3-x_2)=1.5-3=-1.5$$
$$r_2=(x_1-x_3)=0-1.5=-1.5$$
$$r_3=(x_2-x_1)=3-0=3$$

Substitute the above values in the matrix equation

$$[B]=\frac{1}{2A}\begin{bmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & -1.5 & 0 & -1.5 & 0 & 3\\ -1.5 & -4 & -1.5 & 4 & 3 & 0\end{bmatrix} \quad [B]=\frac{1}{2*6}\begin{bmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & -1.5 & 0 & -1.5 & 0 & 3\\ -1.5 & -4 & -1.5 & 4 & 3 & 0\end{bmatrix}$$

$$[B]=\frac{1}{2*6}\begin{bmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & -1.5 & 0 & -1.5 & 0 & 3\\ -1.5 & -4 & -1.5 & 4 & 3 & 0\end{bmatrix} \quad [B]=\frac{1}{12}\begin{bmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & -1.5 & 0 & -1.5 & 0 & 3\\ -1.5 & -4 & -1.5 & 4 & 3 & 0\end{bmatrix}$$

We know that Stress – strain relationshipmatrix for plane stress is

$$\left[D\right]=\frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix} \qquad \left[D\right]=\frac{2*10^5}{1-(0.3)^2}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & \frac{1-0.3}{2}\end{bmatrix}$$

$$\left[D\right]=219.78*10^3\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.3\end{bmatrix}$$

$$[D][B] = 219.78*10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} * \frac{1}{12} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & -1.5 & 0 & 3 \\ -1.5 & -4 & -1.5 & 4 & 3 & 0 \end{bmatrix}$$

$$[D][B] = 18.32*10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & -1.5 & 0 & 3 \\ -1.5 & -4 & -1.5 & 4 & 3 & 0 \end{bmatrix}$$

$$[D][B] = 18.32*10^3 \begin{bmatrix} -4 & -0.45 & 4 & -0.45 & 0 & 0.9 \\ -1.2 & -1.5 & 1.2 & -1.5 & 0 & 3 \\ -0.525 & -1.4 & -0.525 & 1.4 & 1.05 & 0 \end{bmatrix}$$

We know that

$$[B] = \frac{1}{12} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & -1.5 & 0 & 3 \\ -1.5 & -4 & -1.5 & 4 & 3 & 0 \end{bmatrix} \quad [B]^T = \frac{1}{12} \begin{bmatrix} -4 & 0 & -1.5 \\ 0 & -1.5 & -4 \\ 4 & 0 & -1.5 \\ 0 & -1.5 & 4 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

$$[B]^T [D][B] = \frac{1}{12} \begin{bmatrix} -4 & 0 & -1.5 \\ 0 & -1.5 & -4 \\ 4 & 0 & -1.5 \\ 0 & -1.5 & 4 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} 18.32*10^3 \begin{bmatrix} -4 & -0.45 & 4 & -0.45 & 0 & 0.9 \\ -1.2 & -1.5 & 1.2 & -1.5 & 0 & 3 \\ -0.525 & -1.4 & -0.525 & 1.4 & 1.05 & 0 \end{bmatrix}$$

$$[B]^T [D][B] = 1.526*10^3 \begin{bmatrix} 16.78 & 3.9 & -15.21 & -0.3 & -1.57 & -3.6 \\ 3.9 & 7.85 & 0.3 & -3.35 & -4.2 & -4.5 \\ -15.21 & 0.3 & 16.78 & -3.9 & -1.578 & 3.6 \\ -0.3 & -3.35 & -3.9 & 7.85 & 4.2 & -4.5 \\ -1.57 & -4.2 & -1.57 & 4.2 & 3.15 & 0 \\ -3.6 & -4.5 & 3.6 & -4.5 & 0 & 9 \end{bmatrix}$$

Substitute $[B]^T[D][B]$ and A, t values in equation

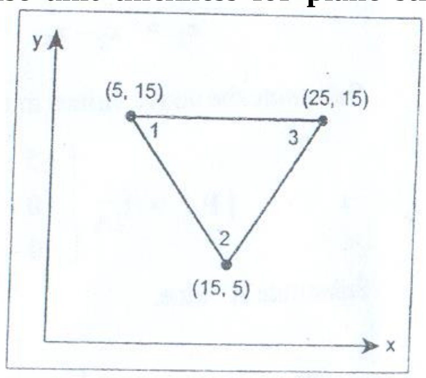
$$[k]=1.526*10^3\begin{bmatrix}16.78 & 3.9 & -15.21 & -0.3 & -1.57 & -3.6\\ 3.9 & 7.85 & 0.3 & -3.35 & -4.2 & -4.5\\ -15.21 & 0.3 & 16.78 & -3.9 & -1.578 & 3.6\\ -0.3 & -3.35 & -3.9 & 7.85 & 4.2 & -4.5\\ -1.57 & -4.2 & -1.57 & 4.2 & 3.15 & 0\\ -3.6 & -4.5 & 3.6 & -4.5 & 0 & 9\end{bmatrix}*10*6N/mm$$

$$[k]=91.6*10^3\begin{bmatrix}16.78 & 3.9 & -15.21 & -0.3 & -1.57 & -3.6\\ 3.9 & 7.85 & 0.3 & -3.35 & -4.2 & -4.5\\ -15.21 & 0.3 & 16.78 & -3.9 & -1.578 & 3.6\\ -0.3 & -3.35 & -3.9 & 7.85 & 4.2 & -4.5\\ -1.57 & -4.2 & -1.57 & 4.2 & 3.15 & 0\\ -3.6 & -4.5 & 3.6 & -4.5 & 0 & 9\end{bmatrix}N/mm$$

10.For the plane strain element shown in fig. the nodal displacement are:

u_1=0.005mm;v_1=0.002mm
u_2=0.0mm; v_2=0.0mm
u_3=0.005mm; v_3=0.0mm. determine the element stresses $\sigma_x,\sigma_y,\tau_{xy}\sigma_1\sigma_2 and\theta p$ *theprincipleangle*, **E=70Gpa, and v=0.3 and use unit thickness for plane strain. All co-ordinates are in mm. (AU-APR/MAY-2010)**

10.



Given: nodal displacements:

B u_1=0.005mm ; v_1=0.002mm
u_2=0.0mm ; v_2=0.0mm
u_3=0.005mm ; v_3=0.0mm
x_1=5mm ; y_1=15mm.
x_2=15mm ; y_2=5mm.
x_3=25mm ; y_3=15mm.
young's modulus, E=70Gpa=70X10^9 pa.=70X10^3 N/mm^2
possion's ratio, v=0.3
unit thickness i=1mm
to find: (1) element stresses

a. Normal stresses , σ_x
b. Normal stresses, σ_y
c. Shear stress, τ_{xy}

d. Maximum mormal stress, σ_1
e. Minimum normal stress, σ_2
f. Principal angle, $\o_P$

Solution:

We know that,

Area of the element, $A=\frac{1}{2A}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}=\frac{1}{2A}\begin{vmatrix} 1 & 15 & 15 \\ 1 & 15 & 5 \\ 1 & 25 & 15 \end{vmatrix}$

$=\frac{1}{2}X\left[1X(15X15-5X25)-5(1X15X\ -5X1)+15(1X25-15X1)\right]$ A=100mm^2

Strain-displacement, $[B]=\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

Where, $q_1=y_2-y_3=5-15=-10$
$q_2=y_3-y_1=15-15=0$
$q_3=y_1-y_2=15-5=10$
$r_1=x_3-x_2=25-15=10$
$r_2=x_1-x_3=5-25=-20$
$r_3=x_2-x_1=15-5=10$

substitute the above values in equation

$$[B]=\frac{1}{2A}\begin{bmatrix} -10 & 0 & q_2 & 0 & 10 & 0 \\ 0 & 10 & 0 & -20 & 0 & 10 \\ 10 & -10 & -20 & 0 & 10 & 10 \end{bmatrix}$$

Substitute area, a value

$$[B]=\frac{1}{2X100}\begin{bmatrix} -10 & 0 & q_2 & 0 & 10 & 0 \\ 0 & 10 & 0 & -20 & 0 & 10 \\ 10 & -10 & -20 & 0 & 10 & 10 \end{bmatrix}$$

We know that,

Stress-strain relationship matrix [D] for plane strain problem is,

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

$$[D] = \frac{70X10^3}{(1+0.3)(1-2X0.3)}\begin{bmatrix} 1-0.3 & 0.3 & 0 \\ 0..3 & 1-0.3 & 0 \\ 0 & 0 & \frac{1-2X0.3}{2} \end{bmatrix}$$

$$[D] = 134.615X10^3\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0..3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$[D] = 134.615X10^3\,X0.2\begin{bmatrix} 3.5 & 1.5 & 0 \\ 1.5 & 3.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D] = 26.923X10^3\begin{bmatrix} 3.5 & 1.5 & 0 \\ 1.5 & 3.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D][B] = 26.923X10^3\begin{bmatrix} 3.5 & 1.5 & 0 \\ 1.5 & 3.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{2X100}\begin{bmatrix} -10 & 0 & q_2 & 0 & 10 & 0 \\ 0 & 10 & 0 & -20 & 0 & 10 \\ 10 & -10 & -20 & 0 & 10 & 10 \end{bmatrix}$$

$$[D][B] = 134.615\begin{bmatrix} -35 & 15 & 0 & -30 & 35 & 15 \\ -15 & 35 & 0 & -70 & 15 & 35 \\ 10 & -10 & -20 & 0 & 10 & 10 \end{bmatrix}$$

We know that,

Stress, {σ}=[D] [B] {u}

$$=[D]\ [B]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = 134.615\begin{bmatrix} -35 & 15 & 0 & -30 & 35 & 15 \\ -15 & 35 & 0 & -70 & 15 & 35 \\ 10 & -10 & -20 & 0 & 10 & 10 \end{bmatrix} X \begin{Bmatrix} 0.005 \\ 0.002 \\ 0.0 \\ 0.0 \\ 0.005 \\ 0.0 \end{Bmatrix}$$

$$= 134.615\begin{Bmatrix} -35X0.005 + 15X0.002 + 35X0.005 \\ -15X0.005 + 35X0.002 + 15X0.005 \\ 10X0.005 - 10X0.002 + 10X0.005 \end{Bmatrix} = 134.615\begin{Bmatrix} 0.003 \\ 0.07 \\ 0.08 \end{Bmatrix}$$

$$\sigma = \begin{Bmatrix} 4.038 \\ 9.4223 \\ 10.769 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 4.038 \\ 9.423 \\ 10.769 \end{Bmatrix}$$

a. Normal stresses , σ_x=1.038 N/mm^2
b. Normal stresses, σ_y=9.423 N/mm^2
c. Shear stress, τ_{xy}=10.769 N/mm^2
We know that,

Maximum normal stress, $\sigma_{max}=\sigma_1=\frac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)+\tau^2{}_{xy}}$

$$=\frac{4.038+9.423}{2}+\sqrt{\left(\frac{4.038-9.423}{2}\right)+(10.769)^2}$$

σ_1=17.83 N/mm^2

Minimum normal stress, $\sigma_{max}=\sigma_2=\frac{\sigma_x+\sigma_y}{2}-\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)+\tau^2{}_{xy}}$

$$=\frac{4.038+9.423}{2}-\sqrt{\left(\frac{4.038-9.423}{2}\right)+(10.769)^2}$$

σ_2=-4.369 N/mm^2

we know that,

priciple angle, $\tan 2\,ø_p=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}=\tan^{-1}\left(\frac{2X10.769}{10.038-9.423}\right)$

$2\,ø_p$=-75.96º

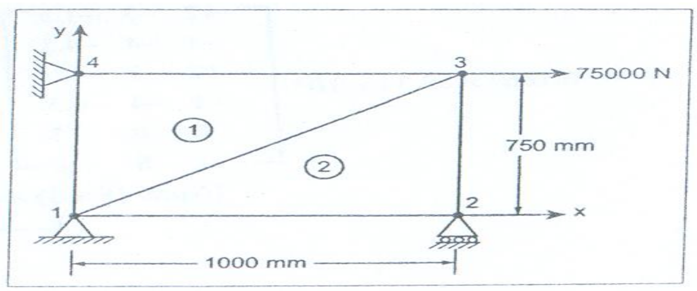
$ø_p$=-37.98º

Result:

a. Normal stresses , σ_x=1.038 N/mm^2
b. Normal stresses, σ_y=9.423 N/mm^2
c. Shear stress, τ_{xy}=10.769 N/mm^2
d. Maximum mormal stress, σ_1=17.83 N/mm^2
e. Minimum normal stress, σ_2=-4.369 N/mm^2
f. Principal angle, $ø_P$=37.98º

11. The two dimensional propped beam shwon in fig. is divided in two CST element. Determine the nodal displacement and element stresses using plate stress costions. Body force is negleced in compression with the external forces.Take: thickness,t=10mm, young's moduls , E=2X10^5 N/mm^2, poisson'sratio,v=0.25

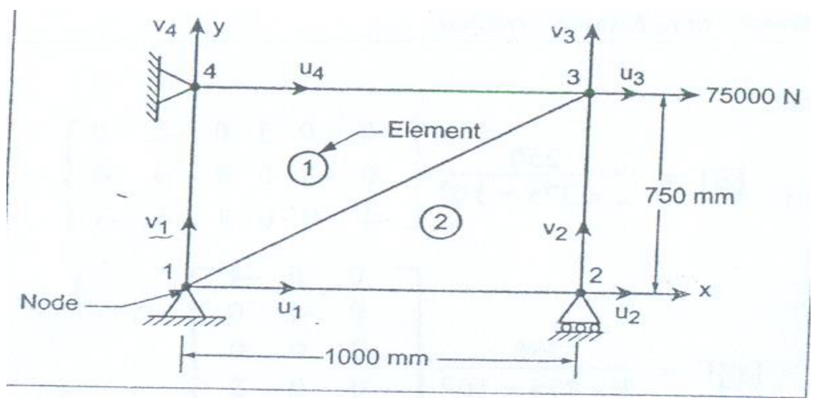
Given:



Thickness, t=10mm

Young's modulus, E=2X10^5 N/mm^2

Poisson's ratio, v=0.25

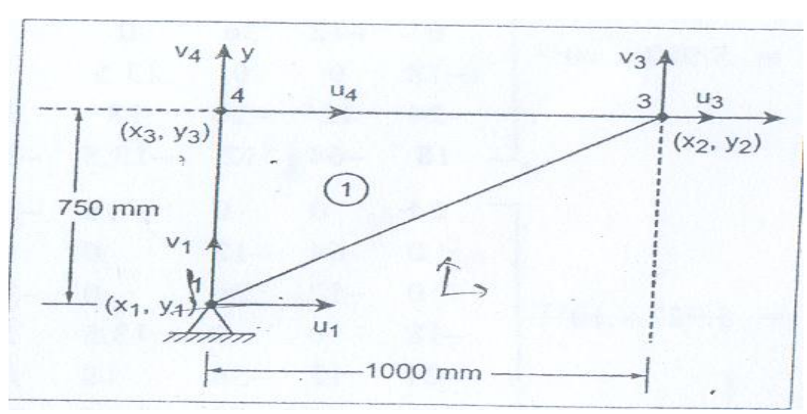


To find: (i) nodal displacement u_1, v_1, u_2,v_2,u_3,v_3 and u_4,v_4

(ii) element stress, $\sigma_{1,}\sigma_2$

Solution:

Consider (1): nodal displacements u_1, v_1, u_2,v_2,u_3,v_3 and u_4,v_4



Take node 1 as origin.

For node 1 : 0,0

For node 2: 1000, 750

For node 3: 0,750

Stiffness matriux, [K]= $[B]^T$ [D] [B] A t

Where, A= area of the triangular element

$$= \frac{1}{2A}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2A}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1000 & 750 \\ 1 & 0 & 750 \end{vmatrix}$$

$$= \frac{1}{2} X 1 X \left(1000 X 750 - 0\right) = \frac{1000 X 750}{2}$$

$$A = 375 X 10^3 mm^2$$

Strain- displacement matrix, [B] = $\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

Where, $q_1=y_2-y_3=750-750=0$

$q_2=y_3-y_1=750-0=750$

$q_3=y_1-y_2=0-750=-750$

$r_1=x_3-x_2=0-1000=-1000$

$r_2=x_1-x_3=0-0=0$

$r_3=x_2-x_1=1000-0=1000$

substitute the above values in equation

$$[B]=\frac{1}{2A}\begin{bmatrix} 0 & 0 & 750 & 0 & -750 & 0 \\ 0 & -1000 & 0 & 0 & 0 & 1000 \\ -1000 & 0 & 0 & 750 & 1000 & -750 \end{bmatrix}$$

Substitute area, a value

$$[B]=\frac{1}{2 X 375 X 10^3}\begin{bmatrix} 0 & 0 & 750 & 0 & -750 & 0 \\ 0 & -1000 & 0 & 0 & 0 & 1000 \\ -1000 & 0 & 0 & 750 & 1000 & -750 \end{bmatrix}$$

$$[B]=\frac{250}{2X375X10^3}\begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}$$

Stress=strain relationship matrix [D] for plane stress problem is,

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \quad [D]=\frac{2X10^5}{\left(1-(0.25)^2\right)}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D]=\frac{2X10^5}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \quad [D]=\frac{2X10^5 X 0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D]=2X10^5 X 0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[B][D]=2X10^5 X 0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} X \frac{250}{2X375X10^3}\begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}$$

$$[B][D]=\frac{250X2X10^5 X 0.2667}{2X375X10^3}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} X \begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}$$

$$[B][D]=17.78\begin{bmatrix} 0+0+0 & 0-4+0 & 12+0+0 & 0+0+0 & -12+0+0 & 0+4+0 \\ 0+0+0 & 0-16+0 & 3+0+0 & 0+0+0 & -3+0+0 & 0+16+0 \\ 0+0-16 & 0+0+0 & 0+0+0 & 0+0+4.5 & 0+0+6 & 0+0-4.5 \end{bmatrix}$$

$$[B][D]=17.78\begin{bmatrix} 0 & -4 & 12 & 0 & -12 & 4 \\ 0 & -16 & 3 & 0 & -3 & 16 \\ -16 & 0 & 0 & 4.5 & 6 & -4.5 \end{bmatrix}$$

We know that,

$$[B]=\frac{250}{2X375X10^3}\begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}[B]^T=\frac{250}{2X375X10^3}\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \\ -3 & 0 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

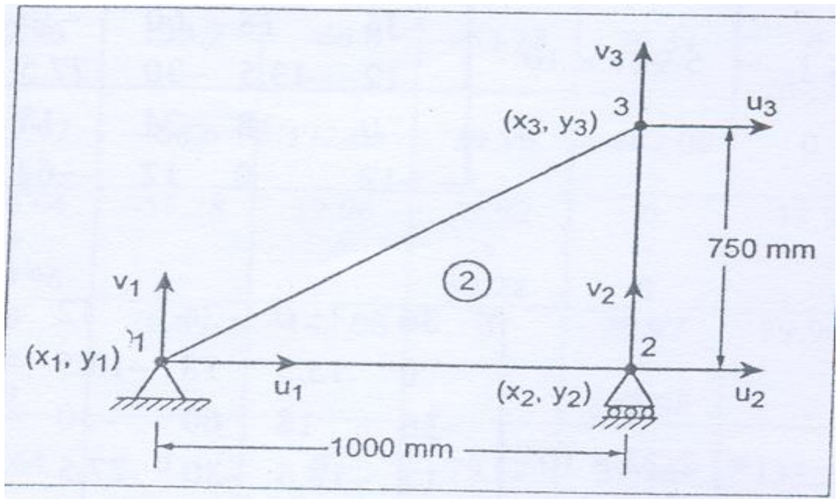
$[B]^T[D]$

$$\frac{250}{2X375X10^3}\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \\ -3 & 0 & 4 \\ 0 & 4 & -3 \end{bmatrix} X \frac{250}{2X375X10^3}\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \\ -3 & 0 & 4 \\ 0 & 4 & -3 \end{bmatrix} X17.78\begin{bmatrix} 0 & -4 & 12 & 0 & -12 & 4 \\ 0 & -16 & 3 & 0 & -3 & 16 \\ -16 & 0 & 0 & 4.5 & 6 & -4.5 \end{bmatrix}$$

$$[K]=5.927\,X10_{-3}\begin{bmatrix} 24 & 0 & 0 & -18 & -24 & 18 \\ 0 & 64 & -12 & 0 & 12 & -64 \\ 0 & -12 & 36 & 0 & -36 & 12 \\ -18 & 0 & 0 & 13.5 & 18 & -13.5 \\ -24 & 12 & -36 & 18 & 60 & -30 \\ 18 & -64 & 12 & -13.5 & -30 & 77.5 \end{bmatrix}$$

$$stiffnessmatri,[k]=1X10^4\begin{bmatrix} 53.28 & 0 & 0 & -39.96 & -53.28 & 39.96 \\ 0 & 142.08 & -26.64 & 0 & 26.64 & -142.08 \\ 0 & -26.64 & 79.92 & 0 & -79.92 & 26.64 \\ -39.96 & 0 & 0 & 29.97 & 39.96 & -29.97 \\ -53.28 & 26.64 & -79.92 & 39.96 & 133.2 & -66.6 \\ 39.96 & -142.08 & 26.64 & -29.97 & -66.6 & 172.05 \end{bmatrix}$$

Consider (2): nodal displacements u_1, v_1, u_2,v_2,u_3,v_3 and u_4,v_4



Take node 1 as origin.

For node 1 : 0,0

For node 2: 1000, 750

For node 3: 0,750

Stiffness matriux, [K_2]= [B]^T [D] [B] A t

Where, A= area of the triangular element

$$= \frac{1}{2A}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A= \frac{1}{2A}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1000 & 0 \\ 1 & 1000 & 750 \end{vmatrix}$$

$$= \frac{1}{2} X 1 X \left(1000 X 750 - 0\right) = \frac{1000 X 750}{2}$$

$$= A = 375 X 10^3 mm^2$$

Strain- displacement matrix, [B] = $\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

Where, $q_1=y_2-y_3=0-750=-750$

$q_2=y_3-y_1=750-0=750$
$q_3=y_1-y_2=0-0=0$
$r_1=x_3-x_2=1000-1000=0$
$r_2=x_1-x_3=0-1000=-1000$
$r_3=x_2-x_1=1000-0=1000$

substitute the above values in equation

$$[B]=\frac{1}{2A}\begin{bmatrix} -750 & 0 & 750 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1000 & 0 & 1000 \\ 0 & -750 & -1000 & 750 & 1000 & 0 \end{bmatrix}$$

Substitute area, a value

$$[B]=\frac{1}{2X375X10^3}\begin{matrix} -750 & 0 & 750 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1000 & 0 & 1000 \\ 0 & -750 & -1000 & 750 & 1000 & 0 \end{matrix}$$

$$[B]=\frac{250}{2X375X10^3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

Stress=strain relationship matrix [D] for plane stress problem is,

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \quad [D]=\frac{2X10^5}{\left(1-(0.25)^2\right)}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D]=\frac{2X10^5}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \quad [D]=\frac{2X10^5 X0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D]=2X10^5 X0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[B][D]=2X10^5 X0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} X \frac{250}{2X375X10^3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

$$[B][D] = 17.78\begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix}$$

We know that,

$$[B] = \frac{250}{2X375X10^3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix} [B]^T = \frac{250}{2X375X10^3}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 3 & 0 & -4 \\ 0 & -4 & 3 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$[B]^T[D]\,[B] = \frac{250}{2X375X10^3}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 3 & 0 & -4 \\ 0 & -4 & 3 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} X17.78\begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix}$$

$$[K] = 5.927\,X10^{-3}\begin{bmatrix} 36 & 0 & -36 & 12 & 0 & 12 \\ 0 & 13.5 & 18 & -13.5 & 18 & 0 \\ -36 & 18 & 60 & -30 & -24 & 12 \\ 12 & -13.5 & -30 & 77.5 & 18 & -64 \\ 0 & -18 & -24 & 18 & 24 & 0 \\ -12 & 0 & 12 & -64 & 0 & 64 \end{bmatrix}$$

$$stiffnessmatri,[k] = 1X10^4\begin{bmatrix} 79.92 & 0 & -79.92 & 26.64 & 0 & -26.64 \\ 0 & 29.97 & 39.96 & -29.97 & -39.96 & 0 \\ -79.92 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 \\ 26.64 & -29.97 & -66.6 & 172.05 & 39.96 & -142.08 \\ 0 & -39.96 & -53.28 & 39.96 & 53.28 & 0 \\ -26.64 & 0 & 26.64 & -142.08 & 0 & 142.08 \end{bmatrix}$$

Global stiffness matrix [K]

Assemble the stiffness matrix equation

Global stiffness matrix, $[k]=1x10^4$

U1	V1	U2	V2	U3	V3	U4	V4	
53.28+ 79.92	0+0	-79.92	26.64	0+0	-39.96 -26.64	-53.28	39.96	U1
0+0	+142.08 +29.97	39.96	-29.97	-26.64+ -39.96	0+0	26.64	-142.28	V1
-79.92	39.96	133.2	-66.6	-53.26	26.64	0	0	U2
26.64	-29.97	-66.6	172.06	39.96	-142.08	0	0	V2
0+0	-26.64+ -39.96	-53.28	39.96	79.92 +53.28	0+0	-79.92	26.64	U3
- 39.96+ -26.64	0+0	26.64	-142.28	0+0	29.97+1 42.08	39.96	-29.96	V3
-53.28	26.64	0	0	-79.92	39.96	133.2	-66.6	U4
39.96	-142.08	0	0	26.64	-29.97	-66.6	172.05	V4

Global stiffness matrix $[k]=1X10^4$

$$
\begin{array}{c}
\begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\
\begin{bmatrix}
133.2 & 0 & -79.92 & 26.64 & 0 & -66.6 & -53.28 & 39.96 \\
0 & 172.05 & 39.96 & -29.97 & -66.6 & 0 & 26.64 & -142.08 \\
-79.922 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 & 0 & 0 \\
26.64 & -29.97 & -66.6 & 172.06 & 39.96 & -142.08 & 0 & 0 \\
0 & -66.6 & -53.28 & 39.96 & 133.2 & 0 & -79.92 & 26.64 \\
-66.6 & 0 & 26.64 & -142.08 & 0 & 172.08 & 39.96 & -29.97 \\
-53.28 & 26.64 & 0 & 0 & -79.92 & 39.96 & 133.2 & -66.6 \\
39.96 & -142.08 & 0 & 0 & 26.64 & -29.97 & -66.6 & 172.05
\end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}
$$

we know that, general force equation is {F}=[k] {u}

$$
\begin{Bmatrix} F_{1-X} \\ F_{1-y} \\ F_{2-x} \\ F_{2-y} \\ F_{3-x} \\ F_{3-y} \\ F_{4-x} \\ F_{4-y} \end{Bmatrix}
= 1X10^4
\begin{bmatrix}
133.2 & 0 & -79.92 & 26.64 & 0 & -66.6 & -53.28 & 39.96 \\
0 & 172.05 & 39.96 & -29.97 & -66.6 & 0 & 26.64 & -142.08 \\
-79.922 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 & 0 & 0 \\
26.64 & -29.97 & -66.6 & 172.06 & 39.96 & -142.08 & 0 & 0 \\
0 & -66.6 & -53.28 & 39.96 & 133.2 & 0 & -79.92 & 26.64 \\
-66.6 & 0 & 26.64 & -142.08 & 0 & 172.08 & 39.96 & -29.97 \\
-53.28 & 26.64 & 0 & 0 & -79.92 & 39.96 & 133.2 & -66.6 \\
39.96 & -142.08 & 0 & 0 & 26.64 & -29.97 & -66.6 & 172.05
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}
$$

Applying boundary conditions

Node 1, and node 4 are fixed. So, u_1, v_1, and u_4,v_4 are =0

1. Node 2, is moving in x direction, so, $u_2=0$ but, $v_2=0$
2. At node 3, a point load of 75000N is acting in x direction, so, F_{3-x}=75,000N
3. Body force is neglected. So, the remaining forces are zero.
4.

5. $$\begin{matrix} F_{1-X} \\ F_{1-y} \\ F_{2-x} \\ F_{2-y} \\ F_{3-x} \\ F_{3-y} \\ F_{4-x} \\ F_{4-y} \end{matrix} = 0$$

Substitute the above values in equation

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 75X10^3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 1X10^4 \begin{bmatrix} 133.2 & 0 & -79.92 & 26.64 & 0 & -66.6 & -53.28 & 39.96 \\ 0 & 172.05 & 39.96 & -29.97 & -66.6 & 0 & 26.64 & -142.08 \\ -79.922 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 & 0 & 0 \\ 26.64 & -29.97 & -66.6 & 172.06 & 39.96 & -142.08 & 0 & 0 \\ 0 & -66.6 & -53.28 & 39.96 & 133.2 & 0 & -79.92 & 26.64 \\ -66.6 & 0 & 26.64 & -142.08 & 0 & 172.08 & 39.96 & -29.97 \\ -53.28 & 26.64 & 0 & 0 & -79.92 & 39.96 & 133.2 & -66.6 \\ 39.96 & -142.08 & 0 & 0 & 26.64 & -29.97 & -66.6 & 172.05 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ v_3 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation u_1, v_1, u_2,v_2,u_3,v_3 and u_4,v_4=0 so, delete the corresponding row and column of [K] matrix. Hence the equation reduces to

$$\begin{Bmatrix} 0 \\ 75X10^3 \\ 0 \end{Bmatrix} = 1X10^4 \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ -53.28 & 133.2 & 0 \\ 26.64 & 0 & 172.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 7.5 \\ 0 \end{Bmatrix} = \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ -53.28 & 133.2 & 0 \\ 26.64 & 0 & 172.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 18.75 \\ 0 \end{Bmatrix} = \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ -53.28 & 133.2 & 0 \\ 26.64 & 0 & 172.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 18.75 \\ 18.75 \end{Bmatrix} = \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ 0 & 279.72 & 26.64 \\ 0 & 0 & -4349.81 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$-4349.81\ v_3=18.75$

$V_3=-0.00431$mm

$279.72u_3+26.64v_3=18.75$

$279.72u_3+26.64X(-0.00431)=18.75$

$u_3=0.067$mm

$u_2=0.02766$mm

nodal displacements:

$u_1=0$mm; $v_1=0$mm

$u_2=0.02766$mm; $v_2=0$mm

$u_3=0.067$mm ,$v_3=-0.00431$

$u_4=0$mm; $v_4=0$mm

stress in each element:

wekow that. Stress, $\{\sigma\}=[D]\ [B]\ \{u\}$

for element (1) nodal displacement u_1, v_1, u_3,v_3 and u_4,v_4

$$\text{stress , }\{\sigma\}=17.78\begin{bmatrix} 0 & -4 & 12 & 0 & -12 & 4 \\ 0 & -16 & 3 & 0 & -3 & 16 \\ -6 & 0 & 0 & 4.5 & 6 & 4.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ \\ \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 14.295 \\ 3.574 \\ -0.345 \end{Bmatrix} N/mm^2$$

For element (2)

$$\text{stress , }\{\sigma\}=17.78\begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \\ \\ \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 5.595 \\ 0.2492 \\ 4.196 \end{Bmatrix} N/mm^2$$

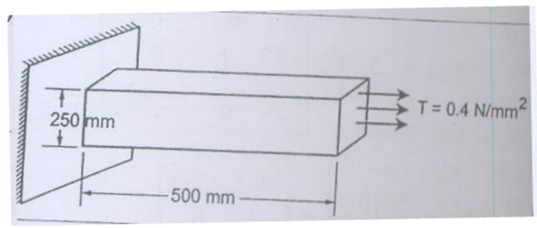
Result:

1. Nodal displacement: u_1=0, v_1=0, u_2 =0.02766,v_2 =0,u_3 =0.067,v_3 =-0.00431 and u_4 =0,v_4 =0
2. Element stresses:

For elements (1) $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 14.295 \\ 3.574 \\ -0.345 \end{Bmatrix} N/mm^2$

For elements (2) $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 5.595 \\ 0.2492 \\ 4.196 \end{Bmatrix} N/mm^2$

12. A thin plate is subjected to surface traction as shown in fig. calculate the global stiffness matrix. (Nov/Dec 2011, April/May 2010)



Take: t=25mm, young's moduls , E=2X10^5 N/mm^2, poisson'sratio,v=0.30.

assume plane stress condition.

Given :

Thickness, t:25mm

Young's modulus, E=2X10^5 N/mm^2

Poisson' ratio, v=0.30

Breadth=250mm

Length, l=500mm

Tensile surface traction, T=0.4 N/mm^2

The tensile surface traction is converted into nodl force.

F=1/2 TA=1/2 X T X (bxt)

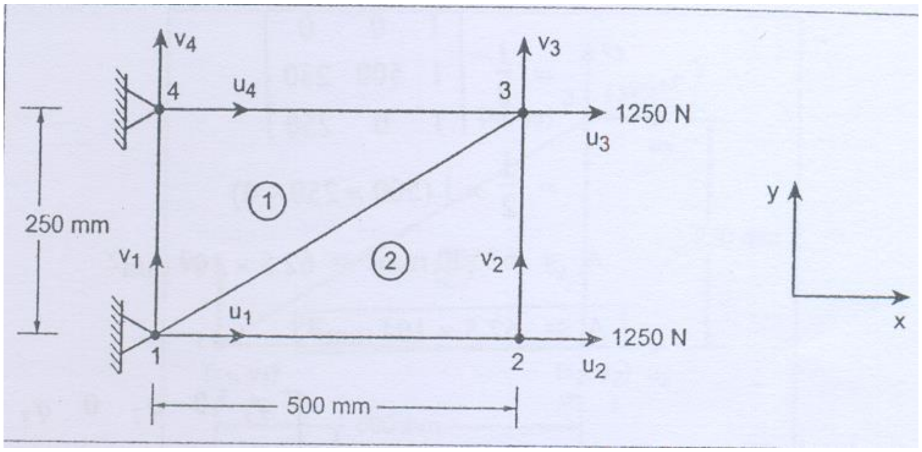
=1/2X0.4X250X25

Nodal force, F=1250N

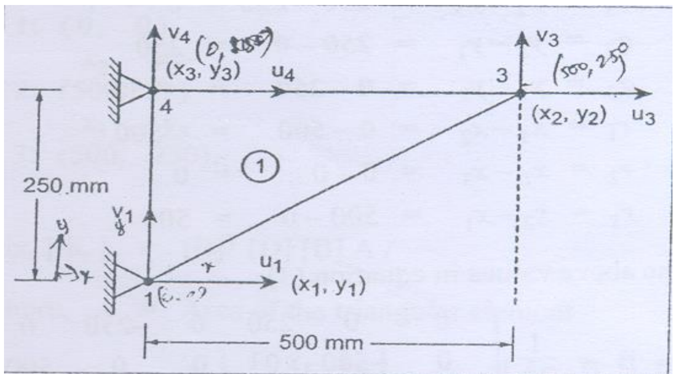
To find:

Stiffness matrix [k]

Solution:



Consider (1): nodal displacements u_1, v_1, u_2,v_2,u_3,v_3 and u_4,v_4



Take node 1 as origin.

For node 1 : (0,0)

For node 2: (500, 250)

For node 3: (0,250)

Stiffness matriux, [K]= $[B]^T$ [D] [B] A t

Where, A= area of the triangular element

$$= \frac{1}{2A}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \frac{1}{2A}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 500 & 250 \\ 1 & 0 & 250 \end{vmatrix} = \frac{1}{2} X 1 X \left(500 X 250 - 0\right) \quad A = 62.5 X 10^3 mm^2$$

Strain- displacement matrix, [B] = $\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

Where, $q_1=y_2-y_3=250-250=0$
$q_2=y_3-y_1=250-0=250$
$q_3=y_1-y_2=0-250=-250$
$r_1=x_3-x_2=0-500=-500$
$r_2=x_1-x_3=0-0=0$
$r_3=x_2-x_1=500-0=500$

Substitute the above values in equation

$$[B]=\frac{1}{2A}\begin{bmatrix} 0 & 0 & 500 & 0 & -250 & 0 \\ 0 & -500 & 0 & 0 & 0 & 500 \\ -500 & 0 & 0 & 250 & 500 & -250 \end{bmatrix}$$

Substitute area, a value

$$[B]=\frac{1}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & 500 & 0 & -250 & 0 \\ 0 & -500 & 0 & 0 & 0 & 500 \\ -500 & 0 & 0 & 250 & 500 & -250 \end{bmatrix}$$

$$[B]=\frac{250}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

Stress=strain relationship matrix [D] for plane stress problem is,

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \quad [D]=\frac{2X10^5}{\left(1-(0.3)^2\right)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D]=\frac{2X10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$[B][D]=\frac{2X10^5}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} X \frac{250}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

$$[B][D]=439.56\begin{bmatrix} 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \end{bmatrix}$$

We know that,

$$[B]=\frac{250}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} [B]^T=\frac{250}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$[B]^T[D][B]=$

$$\frac{250}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} X \frac{2X10^5}{\left(1-(0.3)^2\right)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix} X \frac{250}{2X62.5X10^3}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

$$[K]=1373.59X10^3\begin{bmatrix} 1.4 & 0 & 0 & -0.7 & -1.4 & 0.7 \\ 0 & 4 & -0.6 & 0 & 0.6 & -4 \\ 0 & -0.6 & 1 & 0 & -1 & 0.6 \\ -0.7 & 0 & 0 & 0.35 & 0.7 & -0.35 \\ 1.4 & 0.6 & -1 & 0.7 & 2.4 & -1.3 \\ 0.7 & -4 & 0.6 & -0.35 & -1.3 & 4.35 \end{bmatrix}$$

$$stiffnessmatri,[k]=1X10^3\begin{bmatrix} 1923.08 & 0 & 0 & -961.513 & -1923.026 & 961.513 \\ 0 & 5494.36 & -824.154 & 0 & 824.154 & -5494.36 \\ 0 & -824.24 & 1373.59 & 0 & -1373.59 & 824.24 \\ -961.513 & 0 & 0 & 480.7565 & 961.513 & -480.7565 \\ -1923.026 & 84.24 & -1373.59 & 961.513 & 3296.616 & -1785.6697 \\ 961.513 & -5494.36 & 824.154 & -480.756 & -1785.67 & 5975.1165 \end{bmatrix}$$

Consider (2): nodal displacements u_1, v_1, u_2,v_2,u_3,v_3 and u_4,v_4

Take node 1 as origin.

For node 1 : (0,0)

For node 2: (500, 0)

For node 3: (500,250)

Stiffness matriux, $[K_2]= [B]^T [D] [B] A t$

Where, A= area of the triangular element

$$=\frac{1}{2A}\begin{vmatrix}1 & x_1 & y_1\\1 & x_2 & y_2\\1 & x_3 & y_3\end{vmatrix}\ \frac{1}{2A}\begin{vmatrix}1 & 0 & 0\\1 & 500 & 0\\1 & 500 & 250\end{vmatrix} \begin{matrix}=\frac{1}{2}X1X\left(500X250-0\right)=\frac{500X250}{2}\\ A=62.55X10^3 mm^2\end{matrix}$$

Strain- displacement matrix, $[B] = \frac{1}{2A}\begin{bmatrix}q_1 & 0 & q_2 & 0 & q_3 & 0\\0 & r_1 & 0 & r_2 & 0 & r_3\\r_1 & q_1 & r_2 & q_2 & r_3 & q_3\end{bmatrix}$

Where, $q_1=y_2-y_3=0-250=-250$ $q_2=y_3-y_1=250-0=250$ $q_3=y_1-y_2=0-0=0$

$r_1=x_3-x_2=500-500=0$ $r_2=x_1-x_3=0-500=-500$ $r_3=x_2-x_1=500-0=500$

Substitute the above values in equation

$$[B]=\frac{1}{2A}\begin{bmatrix}-250 & 0 & 250 & 0 & 0 & 0\\0 & 0 & 0 & -500 & 0 & 500\\0 & -250 & -500 & 250 & 1000 & 0\end{bmatrix}$$

Substitute area, a value $[B]=\frac{1}{2X62.5X10^3}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & -2 & 0 & 2\\0 & -1 & -2 & 1 & 2 & 0\end{bmatrix}$

Stress=strain relationship matrix [D] for plane stress problem is,

$$\left[D\right]=\frac{E}{\left(1-v^2\right)}\begin{bmatrix}1 & v & 0\\v & 1 & 0\\0 & 0 & \frac{1-v}{2}\end{bmatrix}\ \left[D\right]=\frac{2X10^5}{\left(1-\left(0.3\right)^2\right)}\begin{bmatrix}1 & 0.3 & 0\\0.3 & 1 & 0\\0 & 0 & \frac{1-0.3}{2}\end{bmatrix}\ \left[D\right]=\frac{2X10^5}{0.91}\begin{bmatrix}1 & 0.3 & 0\\0.3 & 1 & 0\\0 & 0 & 0.35\end{bmatrix}$$

$$\left[B\right]\left[D\right]=\frac{2X10^5}{0.91}\begin{bmatrix}1 & 0.3 & 0\\0.3 & 1 & 0\\0 & 0 & 0.35\end{bmatrix}X\ \frac{250}{2X62.5X10^3}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & -2 & 0 & 2\\0 & -1 & -2 & 1 & 2 & 0\end{bmatrix}$$

$$[B][D]=439.56\begin{bmatrix}-1 & 0 & 1 & -0.6 & 0 & 0.6\\ -3 & 0 & 0.3 & -2 & 0 & 2\\ 0 & -0.35 & -0.7 & 0.35 & 0.7 & 0\end{bmatrix}$$

$$[B]=\frac{250}{2X62.5X10^{3}}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0 & -1 & -2 & 1 & 2 & 0\end{bmatrix}[B]^{T}=2X10^{-3}\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 1 & 0 & -2\\ 0 & -2 & 1\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}$$

We know that,

$$[B]^{T}[D]\,[B]=2X10^{-3}X\,439.96\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 1 & 0 & -2\\ 0 & -2 & 1\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}X\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0 & -1 & -2 & 1 & 2 & 0\end{bmatrix}$$

$$[K]=0.8791\begin{bmatrix}1 & 0 & -1 & 0.6 & 0 & -0.6\\ 0 & 0.35 & 0.7 & -0.35 & -0.7 & 0\\ -1 & 0.7 & 2.4 & -1.3 & -1.4 & 0.6\\ 0.6 & -0.35 & -1.3 & 4.35 & 0.7 & -4\\ 0 & -0.7 & -1.4 & 0.7 & 1.4 & 0\\ -0.6 & 0 & 0.6 & -4 & 0 & 4\end{bmatrix}$$

$$stiffness matri,[k_2]=\begin{bmatrix}1373.59 & 0 & -1373.59 & 824.154 & 0 & -824.154\\ 0 & 480.7565 & 961.513 & -480.7565 & -961.13 & 0\\ -1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154\\ 824.154 & -480.7565 & -1785.667 & 5975.165 & 961.513 & -5494.36\\ 0 & -961.513 & -1923.026 & 961.513 & 1923.026 & 0\\ -824.154 & 0 & 824.154 & -5494.36 & 0 & 5494.36\end{bmatrix}$$

Global stiffness matrix [K]   Assemble the stiffness matrix equation

Global stiffness matrix, [k]=1x10^3

U1	V1	U2	V2	U3	V3	U4	V4	
1923.026+1373.59	0+0	-1373.59	824.154	0+0	-961.513+-824.154	-1923.026	961.513	U1
0+0	5494.36+480.7565	961.513	-480.7565	-824.154+-961.513	0+0	824.154	-5494.36	V1
-1373.59	961.513	3296.616	-1785.667	-1923.026	824.154	0	0	U2
824.24	-480.7565	-1785.667	5975.1165	961.513	-5494.36	0	0	V2
0+0	-824.154+-961.513	0+-1923.026	0+961.513	1373.59+1923.026	0+0	-1373.59	824.154	U3
-961.513+-824.154	0+0	0+824.154	0+-5494.36	0+0	961.513	3296.616	-1785.667	V3
-1923.026	824.154	0	0	-1373.59	961.513	3296.616	-1785.667	U4
961.513	-5494.36	0	0	824.154	-480.7565	-1785.667	-5975.1165	V4

Global stiffness matrix $[k]=1X10^3$

$$
\begin{array}{c}
\begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\
\begin{bmatrix}
3296.616 & 0 & -1373.59 & 824.154 & 0 & -1785.667 & -1923.026 & 961.513 \\
0 & 5975.1165 & 961.513 & -480.7565 & -1785.667 & 0 & 824.154 & -5496.36 \\
-1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154 & 0 & 0 \\
824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -5494.36 & 0 & 0 \\
0 & -1785.667 & -1923.026 & 961.513 & 3296.616 & 0 & -1373.59 & 824.154 \\
-1785.667 & 0 & 824.154 & -5494.36 & 0 & 5975.165 & 961.513 & -480.7565 \\
-1923.026 & 824.24 & 0 & 0 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\
961.513 & -5494.36 & 0 & 0 & 824.154 & -480.7565 & -1785.667 & 5975.1165
\end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}
$$

$$[k] = 1X10^3 X$$

$$
\begin{bmatrix}
3296.616 & 0 & -1373.59 & 824.154 & 0 & -1785.667 & -1923.026 & 961.513 \\
0 & 5975.1165 & 961.513 & -480.7565 & -1785.667 & 0 & 824.154 & -5496.36 \\
-1373.59 & 961.513 & 3296.616 & -1785.667 & -1923.026 & 824.154 & 0 & 0 \\
824.154 & -480.7565 & -1785.667 & 5975.1165 & 961.513 & -5494.36 & 0 & 0 \\
0 & -1785.667 & -1923.026 & 961.513 & 3296.616 & 0 & -1373.59 & 824.154 \\
-1785.667 & 0 & 824.154 & -5494.36 & 0 & 5975.165 & 961.513 & -480.7565 \\
-1923.026 & 824.24 & 0 & 0 & -1373.59 & 961.513 & 3296.616 & -1785.667 \\
961.513 & -5494.36 & 0 & 0 & 824.154 & -480.7565 & -1785.667 & 5975.1165
\end{bmatrix}
$$

THERMAL PROBLEM

Temperature effect:

Distribution of the change in temperature (ΔT) is known as strain. Due to the change in temperature can be considered as an initial strain e_0.

$$\{ e_0 \} = \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix}$$ for general and plain stress problem

Where, ΔT --- change in temperature

α --- co-efficient of thermal expansion

For Plain strain problem

$$\{ e_0\} = (1+\mu)\begin{bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{bmatrix}$$

Where, e_0----initial strain,

μ ----- Poissons ratio

ΔT --- change in temperature

α --- co-efficient of thermal expansion

the stress and strain are releated by the following relation,

$e = D\,(e-e_0)$

$\sigma = D\,(Bu - e_0)$

where, e_0--- initial strain

σ---- stress

D----stress strain relationship matrix

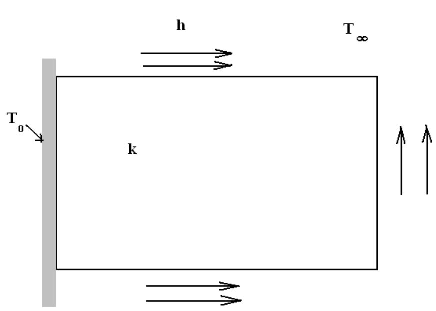
B----strain displacement relationship matrix

u---displacement

Heat transfer in two dimensional elements: the shape function and formulae are similar to that of the CST element.

Governing Equation for 2D Heat transferby conduction and convection

$$k\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right\}-h(T-T_\infty)=0$$



Weak form of the equation

$$\iint \frac{\partial T}{\partial x}\frac{\partial w}{\partial x}dxdy + \iint \frac{\partial T}{\partial y}\frac{\partial w}{\partial y}dxdy + \iint hTw(x,y)dxdy$$

$$= \iint hT_\infty \, w(x,y)dxdy$$

$$K_{ij\,conv} = k\left[\iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}dxdy + \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}dxdy\right]$$

$$\boxed{N_i\,(x,y) \;=\; \frac{1}{2A_e}\,(\alpha_i + \beta_i x + \gamma_i\, y)}$$

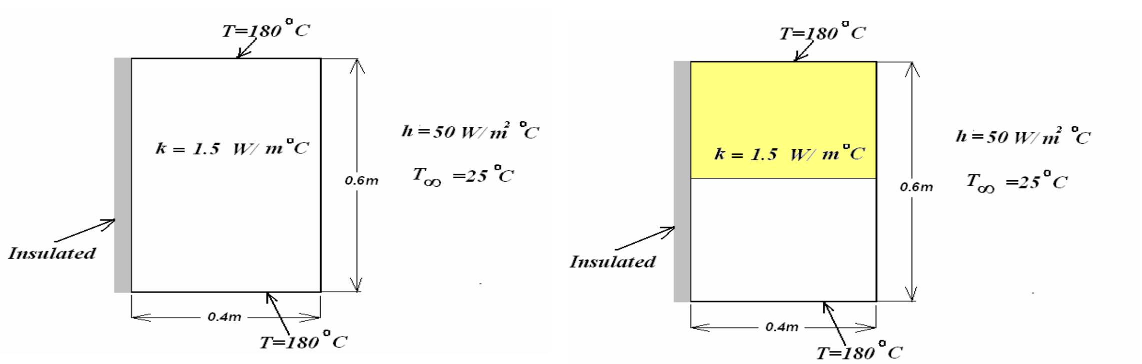
$$[K]^e{}_{conv} = \frac{k}{4A}\begin{bmatrix} \beta_1^2+\gamma_1^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\,\beta_3+\gamma_1\gamma_3 \\ \beta_1\beta_2+\gamma_1\gamma_2 & \beta_2^2+\gamma_2^2 & \beta_2\beta_3+\gamma_2\gamma_3 \\ \beta_1\,\beta_3+\gamma_1\gamma_3 & \beta_2\beta_3+\gamma_2\gamma_3 & \beta_3^2+\gamma_3^2 \end{bmatrix}$$

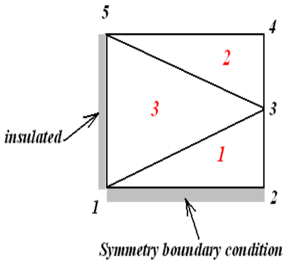
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \frac{hlT_\infty}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$[k]_{conv} = \frac{hpl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

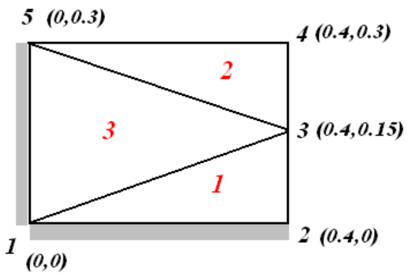
$$\Rightarrow p = 1$$

13. Problem: determine the temperature distribution on the plate





Element	i	j	k
1	1	2	3
2	5	4	3
3	1	3	5



$$[k]^e = \frac{k}{4A}\begin{bmatrix} \beta_1^2+\gamma_1^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\ \beta_3+\gamma_1\gamma_3 \\ \beta_2\beta_1+\gamma_1\gamma_2 & \beta_2^2+\gamma_2^2 & \beta_2\beta_3+\gamma_2\gamma_3 \\ \beta_1\beta_3+\gamma_1\gamma_3 & \beta_2\beta_3+\gamma_2\gamma_3 & \beta_3^2+\gamma_3^2 \end{bmatrix}$$

Element 1 and 2 **Element 3**

$\beta_1 = -0.15$, $\gamma_1 = 0$,	$\beta_1 = 0.15$, $\gamma_1 = -0.4$
$\beta2 = 0.15$, $\gamma_2 = -0.4$	$\beta_2 = 0.3$, $\gamma_2 = 0$
$\beta3 = 0$, $\gamma_3 = 0.4$	$\beta_3 = -0.15$, $\gamma_3 = 0.4$

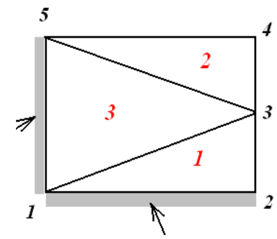
$$[k]^2_{cond} = [k]^1_{cond} = \frac{1.5}{\frac{4}{2}\times 0.4\times 0.15}\begin{bmatrix} 0.0225 & -0.0225 & 0 \\ -0.0225 & 0.1825 & -0.16 \\ 0 & -0.16 & 0.16 \end{bmatrix}$$

$$=10\begin{bmatrix} 0.028125 & -0.028125 & 0 \\ -0.028125 & 0.228125 & -0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}$$

$$=\begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 2.28125 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$[k]^3_{cond} = \frac{1.5}{4\times 2\times \frac{1}{2}\times 0.4\times 0.15}\begin{bmatrix} 0.1825 & -0.045 & -0.1825 \\ -0.045 & 0.09 & -0.045 \\ -0.1825 & -0.045 & 0.1825 \end{bmatrix}$$

$$= \begin{bmatrix} 1.14 & -0.28125 & -0.86 \\ & 0.5625 & -0.28125 \\ & & 1.14 \end{bmatrix}$$



$$[k]_{conv} = \frac{hpl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow p = 1$$

Element	**i**	**j**	**k**
1	1	2	3
2	5	4	3
3	1	3	5

$$[k]^2_{conv} = [k]^1_{conv} = \frac{hl}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.5 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix}$$

$$Q = \frac{hlT_\infty}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 93.75 \\ 93.75 \end{Bmatrix}$$

$$[k]^e{}_{Thermal} = [k]_{condn} + [k]_{conv}$$

$$[k]^1_{th} = [k]^2_{th} = \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 4.78 & -0.75 \\ 0 & -0.75 & 4.5 \end{bmatrix}$$

$$[k]^3_{th} = \begin{bmatrix} 1.14 & -0.28125 & -0.86 \\ -0.28125 & 0.5625 & -0.28125 \\ -0.86 & -0.28125 & 1.14 \end{bmatrix}$$

$$[k]_{th}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = [Q]^G \quad \Rightarrow \quad [Q]^G = \begin{Bmatrix} 0 \\ 93.75 \\ 93.75+93.75 \\ 93.75 \\ 0 \end{Bmatrix}$$

$$[k]^G = \begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix}$$

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = 93.75 \begin{Bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{Bmatrix}$$

Substitute for T_4 & T_5 as 180° and evaluate T_1, T_2, T_3

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & 0.75 & 4.78 & 0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} =$$

$$93.75 \begin{Bmatrix} 0 + 0.86 * 180 \\ 1 \\ 2 + (0.28125 + 0.75) * 180 \\ 1 \\ 0 \end{Bmatrix}$$

T_1 = 124.5°C

T_2 = 34.0°C

T_3 = 45.4°C

STIFFNESS MATRIX FOR BI LINEAR RECTANGULAR ELEMENT

$$N_1 = \left(1 - \frac{x}{2a}\right)\left(1 - \frac{y}{2b}\right) \qquad N_3 = \left(\frac{x}{2a}\right)\left(\frac{y}{2b}\right)$$

$$N_2 = \left(\frac{x}{2a}\right)\left(1 - \frac{y}{2b}\right) \qquad N_4 = \left(1 - \frac{x}{2a}\right)\left(\frac{y}{2b}\right)$$

$$= \frac{k}{6ab} \begin{bmatrix} 2(a^2 + b^2) & a^2 - 2b^2 & -(a^2 + b^2) & (b^2 - 2a^2) \\ a^2 - 2b^2 & 2(a^2 + b^2) & (b^2 - 2a^2) & -(a^2 + b^2) \\ -(a^2 + b^2) & (b^2 - 2a^2) & 2(a^2 + b^2) & (-2b^2 + a^2) \\ (b^2 - 2a^2) & -(a^2 + b^2) & a^2 - 2b^2 & 2(a^2 + b^2) \end{bmatrix}$$

14. Explain about the torsion of non-circular shaft

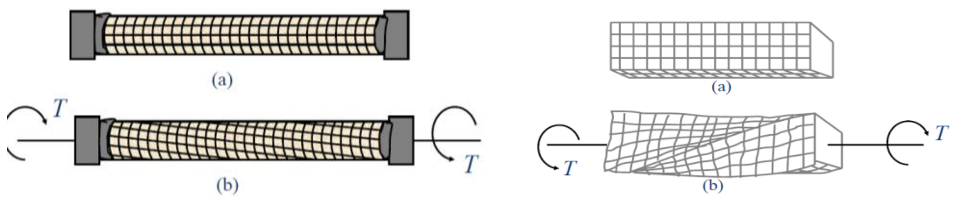
TORSION OF NON-CIRCULAR SHAFT

The first application area is the torsion of Non-Circular sections. The governing differential equation is

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2}+\frac{1}{G}\frac{\partial^2\phi}{\partial y^2}+2\theta = 0$$

where G - shear modulus of the material

θ - is the angle of twist.



The governing equation for the torsion problem is given by

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2}+\frac{1}{G}\frac{\partial^2\phi}{\partial y^2}+2\theta = 0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = -2G\theta$$

$$\tau_{zx} = \frac{\partial\phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial\phi}{\partial x}$$

On the free boundary $\phi = 0$.

Here ϕ - is a stress function

The shear stresses within the shaft are related to the derivatives of ϕ with respect to x and y.

$$\tau_{zx} = \frac{\partial\phi}{\partial y} \quad \text{and} \quad \tau_{zy} = -\frac{\partial\phi}{\partial x}$$

On the free boundary $\phi = 0$. This is the case of a Poisson's Equation

To derive the weak form multiply the governing equation with a weighting function w(x,y)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+2G\theta = 0$$

$$\iint\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+2G\theta\right)w(x,y)\,dxdy = 0$$

On simplifying, the shape function can be expressed as

$$\boxed{N_i\,(x,y) = \frac{1}{2A_e}\,(\alpha_i + \beta_i x + \gamma_i\, y)}$$

$$[K] = \frac{1}{4A}\begin{bmatrix} \beta_1^2+\gamma_1^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\beta_3+\gamma_1\gamma_3 \\ & \beta_1^2+\gamma_2^2 & \beta_2\beta_3+\gamma_2\gamma_3 \\ & & \beta_3^2+\gamma_3^2 \end{bmatrix}$$

Finite element equation for torsional triangular element

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = 2G\theta\frac{A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Where, θ angle o twist, f force vector

QUADRILATERAL ELEMENTS:

Each node is allowed to move in only one direction '+x' in one dimensional problems, but in two dimensional problems, each node is permitted to move in the two directions (i.e) x and y. Each node has two degrees of freedom (Nodal Displacements)

Derivation of shape function for the quadrilateral elements:

Consider a four node rectangular element, the coordinates of the nodes are

Node 1: x_1,y_1Node 2: x_2,y_2Node 3: x_3,y_3Node 4: x_4,y_4,

It has eight unknown displacement degrees of freedom.

The shape functions are

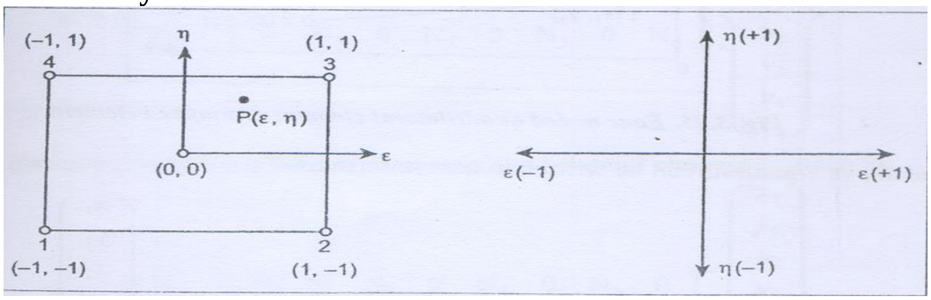
N1 =(1-x/l-y/h+xy/lh), N2 =(x/l-xy/lh) N3 =(xy/lh) N4 =(y/h-xy/lh)

Assembling the equations in matrix form

Displacement form,

$$u = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

15. Derive shape functions for 4 noded rectangular parent element by using natural co-ordinate system and coordinate transformation.



Consider a four noded rectangular element as shown in fig. the parent element is defined in ε and η co-ordinates natural co-ordinates ε is varying from -1 to 1 and η is also varying -1 to 1

We know that,

Shape function value is unity at its own node and its value is zero at other nodes.

At node 1:

(co-ordinates ε=-1, η=-1)

Shape function $N_1=1$ at node 1

$N_1=0$ at node 2, 3, and 4

N_1 has to be in the form oof $N_1=C(1-\varepsilon)(1-\eta)$

Where C is cnsatant

Substitute ε=-1 and η=-1 in equation $N_1=C(1+1)(1+1)$

$N_1=4C$ 1=4C C=1/4

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

At node 2:

(co-ordinates ε=1, η=-1)

Shape function $N_2=1$ at node 2

N_2=0 at node 1, 3, and 4

N_2 has to be in the form oof N_2=C(1+ε) (1-η)

Where C is cnsatant

Substitute ε=1 and η=-1 in equation N_2=C(1+1) (1+1)

N_2=4C 1=4C C=1/4

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

At node 3:

(co-ordinates ε=1, η=1)

Shape function N_3=1 at node 3 N_3=0 at node 1, 2, and 4

N_3 has to be in the form of N_3=C(1+ε) (1+η)

Where C is cnsatant

Substitute ε=1 and η=1 in equation N_3=C(1+1) (1+1)

N_3=4C 1=4C C=1/4

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

At node 4:

(co-ordinates ε=-1, η=1)

Shape function N_4=1 at node4 N_4=0 at node 1, 2, and 3

N_4 has to be in the form of N_4=C(1-ε) (1+η)

Where C is cnsatant

Substitute ε=1 and η=1 in equation N_4=C(1+1) (1+1)

N_3=4C 1=4C C=1/4

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

Consider a point P with coordinate (ε,η) if displacement function u= $\begin{Bmatrix} u \\ v \end{Bmatrix}$ represents to be displacement components of a point located at (ε,η) then,

$u=N_1u_1+N_2\ u_2=N_3\ u_3+N_4\ u_4$ and $v=N_1v_1+N_2\ v_2=N_3v+N_4\ v_4$

it can be in matrix form as,

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

In the isoparametric formulation. For global system, the co0ordinates of the nodal points are (x_1,y_1), (x_2,y_2) ,), (x_3,y_3) and), (x_4,y_4) in order to get mapping the co-ordinate point p is defined as

$x=N_1x_1+N_2\ x_2=N_3\ x_3+N_4\ x_4$ and $y=N_1y_1+N_2\ y_2=N_3y+N_4\ y_4$

The above equations can be written in matrix form as

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

HIGHER ORDER ELEMENTS:

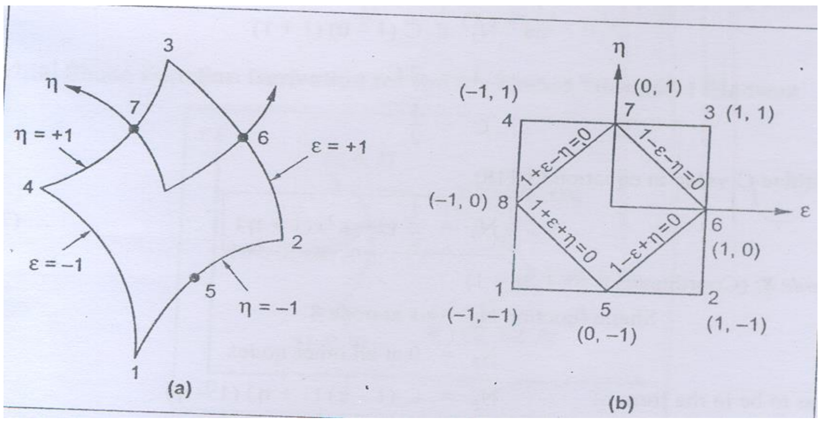
The higher order element can be either a complex or a multiple element.

Higher order elements are nothing but if the interpolation polynomial is the order of two or more elements.

In the higher order elements, some secondary node are introduced in addition to the primary nodes in order to match the number of nodal degrees of freedom with the number of constants in the interpolation polynomial.

16. Derive the shape function for eight noded triangular element.

Consider a eight noded rectangular element. It belongs to the serendipity family of elements. It consists of eight nodes, which are located on the boundary.



$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \qquad N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

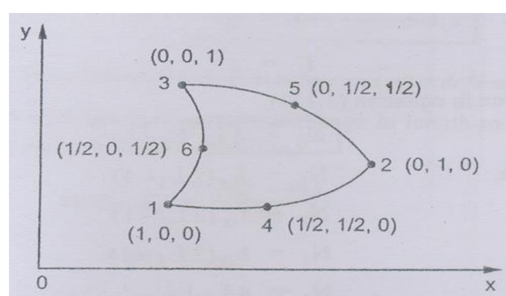
$$N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \qquad N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$N_5 = \frac{1}{2}(1-\xi)(1-\eta)(1+\xi) \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta)(1+\eta)$$

$$N_7 = \frac{1}{2}(1+\xi)(1+\eta)(1-\xi) \qquad N_8 = \frac{1}{2}(1-\xi)(1+\eta)(1-\eta)$$

17. Derive the shape function for six noded triangular element.

Consider a six noded triangular element. It belongs to the serendipity family of elements. It consists of six nodes, which are located on the boundary.



$$N_1 = L_1\left(2L_1 - 1\right) \quad N_2 = L_2\left(2L_2 - 1\right) \qquad N_3 = L_3\left(2L_3 - 1\right)$$

$$N_4 = 4L_1L_2 \qquad N_5 = 4L_2L_3 \qquad N_6 = 4L_1L_3$$

Two Marks Question and Answers.

UNIT-III

1. **What meant by plane stress analysis?**

   Plane stress is defined to be a state of stress in which the normal stress and shear stress directed perpendicular to the plane are assumed to be zero.

2. **Define plane strain analysis. (Nov 2011)**

   Plane strain is defined to be state of strain normal to the xy plane and the shear strains are assumed to be zero.

3. **Write down the stiffness matrix equation for two dimensional CST elements.**

   Stiffness matrix $[K]=[B]^T [\Delta][B]At$

   $[B]^T$ –Strain displacement $[\Delta]$-Stress strain matrix$[B]$-Strain displacement matrix

4. **How do you define two dimensional elements?(AU-NOV/DEC-2010)**
   Two dimensional elements are define by three or more nodes in a two dimensional plane. The basic element useful for two dimensional analyses is the triangular elements. The elements are analyzed in two different axis as x and y axis. The two dimensional elements are used to analyze bar element, beam element and the truss element. The structured analysis is carried out by means of governing equation and the displacement function with its boundary conditions.
5. **What is CST element?**
   Three noded triangular element is known constant strain triangle (CST) which is shown in fig. it has six unknown displacement degree of freedom(u_1 v_1,u_2 v_2,u_3 v_3).the element is called CST because it has a constant strain throughout it.
   Merits:
   Calculation of stiffness matrix is easier.
   Demerits:
   The strain variation within the element is considered as constant. So, the results will be poor.
6. **What is LST element?(AU-NOV/DEC-2011)**
   Linear strain triangular element:
   Six noded triangular element is known as linear strain triangular (LST). it has twelve unknown displacement degree of freedom. The displacement function for the element are quadratic instead of linear as in the CST.the nodes are arise in between the nodes in the CST(constant strain triangular elemnt. the elements are analysed and are a time consuming one.
7. **What is QST element?**
   Ten noded triangular element is known as quadratic strain triangle (QST) it is also called as cubic displacement triangle.they are used to analyse the large structure to find the closest value and to rectify the errors occur due to analysis. increasing the nodes will always gives us the closest or nearest value to the original solution and we may get small differences like tolerance value while we Are diving it as many element.
8. **Write a displacement function equation for CST element.**

Displacement function u= $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$

Where N1, N2, N3 are shape functions.

9. Write a stain-displacement matrix for CST element. (AU-NOV/DEC-2013)

Strain –displacement matrix for CST element is,

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

Where, A= area of the element

$q_1=Y_2-Y_3$; $q_2=y_3-y_1$; $q_3=y_1=y_2$

r_1=x3-x2; $r_2=x_1-x_3$; $r_3=x_2-x_1.$

10. Write down the stiffness equation for two dimensional CST element.

Stiffness matrix,

$$[k] = [B]^T [D]At$$

Where, [B]-strain –displacement matrix

[D]-Stress-strain matrix

A-area of the element

t-thickness of the element.

11. Write down the expression for the shape functions for a constant strain triangularelement. (AU-APR/MAY-2010)

For CST element,

Shape function, $N_1 = \dfrac{p_1+q_1x+r_1y}{2A}$

$N_2 = \dfrac{p_2+q_2x+r_2y}{2A}$

$N_3 = \dfrac{p_3+q_3x+r_3y}{2A}$

Where, $p_1=x_2y_3-x_3$y2

$P_2=x_3y_1-x_1y_3$

$P_3=x_1y_2$-x2y_1

$q_1=Y_2-Y_3$; $q_2=y_3-y_1$; $q_3=y_1=y_2$

$r_1=x_3-x_2$; $r_2=x_1-x_3$; $r_3=x_2-x_1.$

12. Differentiate CST andLST element.

Constant strain triangular element (CST):

CST element is a two dimensional linear element (simplex element). It has only three primary nodes at the corners. The strain is constant throughout the element. The polynomial function for CST element is u=a1+a2x+a3y; v=a4+a5x+a6y.

Linear strain triangular element(LST):

LST is a two dimensional non-linear element (complex element). It has three primary nodes at the corners and three secondary nodes at the midsides. The strain is varying linearly inside the element. The polynomial function for the LST element are u=a1+a2x+a3y+a4x2+a5y2+a6xy; v=a7+a8x+a9y+a10x2+a11y2+a12xy.

13. What is meant by two dimensional scalar variable problem?

Two dimensional scalar variable problem:

If the geometry and material properties of any element are described by two spatial coordinates then that element is referred as two dimensional finite element and in a problem containing that element if the measured parameter is having only one quantity (magnitude only) and not having direction of application then it is referred to as two – dimensional scalar variable problem.

14. Specify the applications of two –dimensional problems.

Applications of two dimensional problems:

1. The plates under bi-axial loading, to find the load, stress, strain and displacement of the plates.
2. The bending of plates to find the load and moment acting on the beam and the displacement function.
3. The temperature distribution on the surface due to heat transfer to find the temperature distribution in the element.

15. Write short noteon finiteelementmodelingoftwodimensionalelement.

Finite element modeling of two dimensional element:

Finite element modeling is the discretization of bigger element of irregular shape into many number of calculatable regular shapes of small sized elements. This process is followed in one dimensional object or element to reform the tapered rod into cylindrical rod. In two dimensional element they are discretised as triangular element to get the regular interpretation of the two dimensional element.

16. Differentiatesimplexandcomplexelements?

Simplex element:

The non-structured problems are elements are discretized in different parameters. If it is discretisized as a triangular element then the element is known to be simplex element.

Complex element:

The complicated structure which are in need or the easy to solve or find the solution some of the element are discretisized as rectangular, quadrilateral, parallelogram such kind of elements are known to be complex element.

17. What are structural and non-structural problems?

Structural problems: In structural problems, displacement at each nodal point is obtained. By

using these displacement solutions, stress and strain in each element can be calculated.
Non-structural problems: In non-structural problems, temperatures or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow, etc., for each element can be calculated.

18. Define shape function. (AU MAY 2008)

In finite element method, field variables with in an element are generally expressed by the following approximate relation:

$\Phi(x, y) = N_1(x, y)\ \phi_1 + N_2(x, y)\ \phi_2 + N_3(x,y)\ \phi 3$

N_1 N_2&N_3 are also called shape functions because they are used to express the geometry or shape of the element.

ϕ_1 ϕ_2 &ϕ_3 are the values of the field variable at the nodes and N_1 N_2&N_3 are the interpolation functions

19. If a displacement field in x direction is given by u: 2 x_2 +4 y_2 + 6xy. Determine the Strain in x direction.

U: $2x_2+4y_2+6xy$

Strain, e $\partial u/\partial x = 4x+6y$

20. What are the ways in which a three dimensional problem can be reduced to a two dimensional approach.

Plane stress: one dimensional is too small when compared to other two dimensions.
Example: Gear thickness is small

Plane strain: one dimensional is too large when compared to other two dimensions.
Example: long pipe [length is long compared to diameter] **Axisymmetric:** geometry is symmetry about the axis.
Example: Cooling tower.

21. What is the purpose of isoparametric elements? (AU DEC 2007)

It is difficult to represent the curved boundaries by straight edge finite elements. A large number of finite elements may be used to obtain reasonable resemblance between original body and the assemblage. In order to overcome this drawback, isoparametric elements are used. i.e., for problem involving curved boundaries, a family to elements is known as "isoparametric elements are used.

22. Give examples for essential (forced or geometric) and non-essential boundary conditions. (AU DEC 2010)

The geometric boundary conditions are displacement, slope, etc. the natural boundary conditions are bending moment, shear force, etc.

23. What are the types of non-linearity? (AU MAY 2010)

i. Non – linearity in material behavior from point to point.
ii. Non – linearity in loading- deformation relation.
iii.Geometric Non – linearity
iv.Change in boundary condition for different loading

24. Define frequency of vibration.

It is the number of cycles described in one second. Unit is Hz.

25. Define Damping ratio.

It is defined as the ratio of actual damping coefficient (C) to the critical damping coefficient (Cc).

Damping ration $\varepsilon = C / C_c = C / 2m\omega_n$

26. What is meant by longitudinal vibrations?

When the particles of the shaft or disc moves parallel to the axis of the shaft, then the vibrations are known as longitudinal vibrations.

27. What is meant by transverse vibrations?

When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, then the vibrations are known as transverse vibrations.

28. Define magnification factor.

The ratio of the maximum displacement of the forced vibration (x_{max}) to eh static deflection under the static force (X_o) is known as magnification factor.

29. Write down the expression of longitudinal vibration of bar element.

Free vibration equation for axial vibration of bar element is,

$[K]\{u\} = \omega^2[m]\{u\}$

Where, u – displacement

[K] – stiffness matrix

ω– Natural frequency

[m] – Mass Matrix

30. Write down the expression of governing equation for free axial vibration of rod.

The governing equation for free axial vibration of a rod is given by,

Where, E – young's modulus,

A – Cross-sectional area

ρ- Density

31. Write down the expression of governing equation for transverse vibration of beam.

The governing equation for free transverse vibration of a beam

Where, E – young's modulus

I – moment of inertia

A – Cross-sectional area

ρ– Density

32. Write down the expression of transverse vibration of beam element.

Free vibration equation for transverse vibration of beam element is,

$[K]\{u\} = \omega^2[m]\{u\}$

Where, [K] = stiffness matrix for beam element

33. What are the types of Eigen value problems?

There are essentially three groups of method of solution,

1. Determinant based methods.
2. Transformation based methods.

3. Vector iteration methods.

34. State the principle of superposition.

It states that for linear systems, the individual responses to several disturbances or driving functions can be superposed on each other to obtain the total response of the system.

35. Define resonance.

When the frequency of external force is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This phenomenon is known as resonance.

36. Define Dynamic Analysis.

When the inertia effect due to the mass of the components is also considered in addition to the externally applied load, then the analysis is called dynamic analysis.

37. What are the methods used for solving transient vibration problems?

There are two methods for solving transient vibration problems. They are:

a) Mode superposition method

b) Direct integration method

38. State the two difference between direct and iterative methods for solving system of equations.

Direct Method

i) It gives exact value.

ii) Simple, take less time.

iii) Determine all the roots at the same time.

Iterative Method

i) It gives only approximate solution.

ii) Time consuming and labourious.

iii) Determine only one root at the time.

39. Define linear dependence and independence of vectors.

Linear dependence : The vectors $X_1, X_2, \ldots X_n$ are said to be linearly dependent if there exist scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ (not all zero) such that,

$\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n = 0$

Independence: The vectors $X_1, X_2, \ldots X_n$ are said to be linearly independent if,

$\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n$ is equal to zero such that

$\lambda_1 = 0, \lambda_2 = \ldots\ldots = \lambda_2$

40. Define Heat transfer.

Heat transfer can be defined as the transmission of energy from one region to another region due to temperature difference.

41. Write down the stiffness matrix equation for one dimensional heat conduction element.

Stiffness matrix, [K] =

Where,

A = area of the element, m^2

K = thermal conductivity, W/mK

l = length of the element, m

42. Write down the expression of shape function, N and temperature function, T for one dimensional heat conduction element.

For one dimensional heat conduction element,
Temperature function, $T = N_1T_1 + N_2T_2$

43. Writedown the finite element equation for one dimensional heat conduction with free end convection.

Finite element equation for one dimensional element heat conduction with free end convection is given by, $T = N_1T_1 + N_2T_2 + N_3T_3$

44. Define path line.

A path line is defined as locus of points through which a fluid particle of fixed identity passes as it moves in space.

45. Define streamline.

A streamline is an imaginary line that connects a series of points in space at a given instant in such a manner that all particles falling on the line at that instant have velocities whose vectors are tangent to the line.

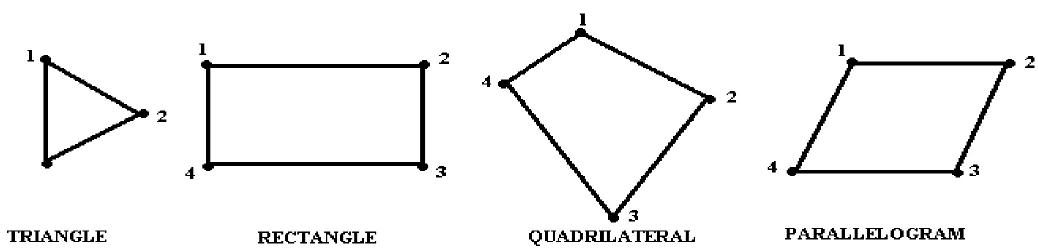
46. Define Inviscid flow.

Ainviscid flow is a frictionless flow characterized by zero viscosity. A viscous flow is one in which the fluid is assumed to have non-zero viscosity.

47. Define two dimensional element.

Twodimensionalelements

Twodimensional elementsaredefinedby threeormorenodesinatwo dimensional plane(i.e.,x,y plane).Thebasicelementusefulfortwodimensional analysisis thetriangular element.



48. Define Plane Stress andPlane Strain

The2delementisextremelyimportantforthePlaneStressanalysisandPlane Strainanalysis.

Plane Stress Analysis:

Itisdefinedtobeastateofstressinwhichthenormalstress(σ)andshear stress (τ)directedperpendicular totheplaneareassumedtobezero.

Plane StrainAnalysis:

Itisdefinedtobeastateofstraininwhichthenormaltothexyplaneand theshearstrainareassumedtobezero

49. Write the expression for TemperatureEffects

Distributionofthechangeintemperature(ΔT)isknownasstrain.Duetothe changeintemperaturecanbeconsideredas aninitialstraine$_0$.

σ=D(Bu- e$_0$)

50. Define variational formulation

Variational formulation refers to the construction of a functional or a variational principle that is equivalent to the governing equations of the problem. It is nothing but the formation I which the governing equations are translated into equivalent weighted integral statements that are not necessarily equivalent to a variational principle.

51. Write the Stress-strain relationship matrix [D] for plane strain problem.

Stress-strain relationship matrix [D] for plane strain problem is,

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

52. Write the Stress-strain relationship matrix [D] for plane stress problem.

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

53. Write the Strain displacement relationship matrix [B] for CST element.

[B]-strain-displacement matrix

$$[B]=\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$q_1 = (y_2 - y_3)$

$q_2 = (y_3 - y_1)$

$q_3 = (y_1 - y_2)$

$r_1 = (x_3 - x_2)$:

$r_2 = (x_1 - x_3)$:

$r_3 = (x_2 - x_1)$:

St.Anne's College of Engineering & Technology,
Department of Mechanical Engineering

Subject Name	**:**	**FINITE ELEMENT ANALYSIS**
Subject code	**:**	**ME8692**
Year	**:**	**IIIrd year**
Semester	**:**	**VIth semester**

UNIT IV
TWO DIMENSIONAL VECTOR VARIABLE PROBLEMS

Equations of elasticity – Plane stress, plane strain and axisymmetric problems – Body forces and temperature effects – Stress calculations - Plate and shell elements.

1. **Explain the equation of elasticity in detail.**

Elasticity

Elasticity is the property of a deformable body due to which the body recovers its original shape uponthe removal of forces causing deformation.

Assumptions made on elasticity:

- Perfectly elastic
- Homogeneous
- Isotropic

Elasticity equations are used for solving structural mechanical problems. There are four basics sets of elasticity equation. They are,

i. Equilibrium equation
ii. Compatibility equation
iii. Strain-displacement relationship equation
iv. Stress-Strain relationship equation

Equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \; ; \; \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + B_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + B_z = 0$$

σ – Stress, τ – Shear Stress, B_x - Body force at X direction,

B_y - Body force at Y direction, B_z - Body force at Z direction.

Compatibility equation

There are six independent compatibility equations, one of which is

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

The other five equations are similarly second order relations.

Strain-displacement relationship equation

$$e_x = \frac{\partial u}{\partial x};\ e_y = \frac{\partial v}{\partial y};\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x};$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

e_x – Strain in X direction, e_y – Strain in Y direction.

γ_{xy} - Shear Strain in XY plane, γ_{xz} - Shear Strain in XZ plane,

γ_{yz} - Shear Strain in YZ plane

Stress-Strain relationship equation

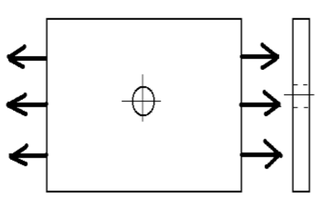
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_x \\ e_x \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

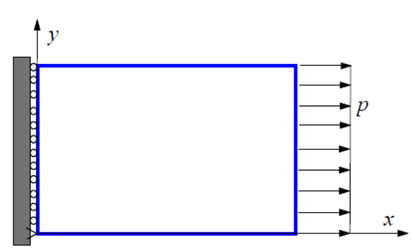
σ – Stress, τ – Shear Stress, E – Young's Modulus, v – Poisson's Ratio,

e – Strain, γ - Shear Strain.

2. **Differentiate plane stress and plane strain.**

PLANE STRESS: - A 3D problem can be reduced to a plane stress condition if it is characterized by very small dimensions in one of the normal directions.A thin plate with a cut out subjected to in-plane loading.Thin plate subjected to in-plane loading





In these cases the stress components σ_z, τ_{xz}, &τ_{yz} are zero and it is assumed that no stress component varies across the thickness. The state of stress is then specified by σ_x, σ_y and τ_{xy} only, (functions of x & y) and is called plane stress. The stress strain relations are given by

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\mu^2}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \tau_{xy}\end{Bmatrix}$$

PLANE STRAIN:-

There exist problems involving very long bodies i.e. a body whose geometry and loading do not vary significantly in the longitudinal direction. Such problems are referred to as plane strain problems.

Some typical examples include a long cylinder such as a tunnel, culvert or buried pipe, a laterally loaded retaining wall, a long earth dam, and a loaded semi-infinite half space such as a strip footing on a soil mass.In all these problems, the dependant variable can be assumed to be functions of only x & y co-ordinates provided that we consider a cross-section some distance away from the two ends.

If we further assume that 'w' the displacement component in the 'z' direction is zero at every cross-section, then the non-zero strain components will be

$$\varepsilon_x = \partial u/\partial x \quad ; \varepsilon_y = \partial v/\partial y \quad ; \gamma_{xy} = \partial u/\partial y + \partial v/\partial x$$

and the strain components ε_z, γ_{xz}, γ_{yz} will vanish. The dependant stress variables are σ_x, σ_y&τ_{xy} and the constitutive relation for an elastic isotropic material is given by

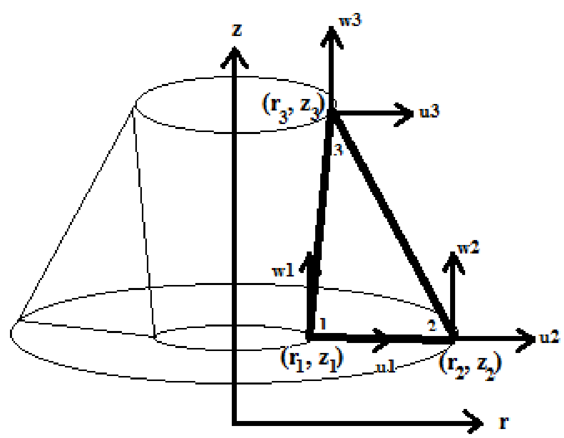
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix}(1-\mu) & \mu & 0\\ \mu & (1-\mu) & 0\\ 0 & 0 & \frac{(1-2\mu)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \tau_{xy}\end{Bmatrix}$$

It is important to note here that only $\varepsilon_z = 0$ but $\sigma_z \neq 0$.

$\varepsilon_z = \sigma_{z/E}$- $\mu/E;\sigma_x - \mu/E;\sigma_y = 0$

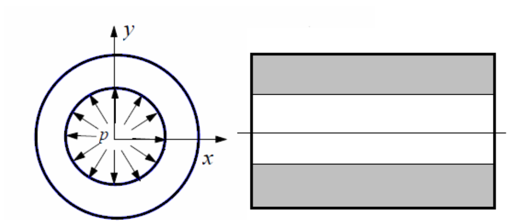
$\therefore \sigma_z = -\mu(\sigma_x+\sigma_y)$

Axisymmetric Elements Most of the three dimensional problems are symmetry about an axis of rotation. Those types of problems are solved by a special two dimensional element called as axisymmetric element.



Many engineering problems involve solids of revolution (axisymmetric solids) subject to axially symmetric loading.

Examples are a circular cylinder loaded by uniform internal or external pressure or other axially symmetric loading as shown in



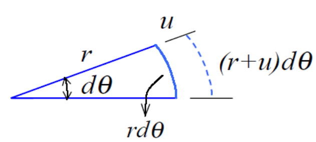
Because of symmetry the stress components are independent of the angular co-ordinate **'θ'** and hence all the derivatives with respect to 'θ' vanish and the components $\gamma_{x\theta}$, $\gamma_{r\theta}$ are zero. The strain displacement relation are given by

$\varepsilon_r = \partial u / \partial x$; $\varepsilon_\theta = u/r$; $\varepsilon_z = \partial w/\partial z$; $\gamma_{rz} = \partial u/\partial z + \partial w/\partial r$

Strains:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \quad (\gamma_{r\theta} = \gamma_{z\theta} = 0)$$



Axisymmetric Formulation

The displacement vector u is given by

$$u(r,z) = \begin{Bmatrix} u \\ w \end{Bmatrix}$$

The stress σ is given by

$$Stress, \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

The strain e is given by

$$Strain, \{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

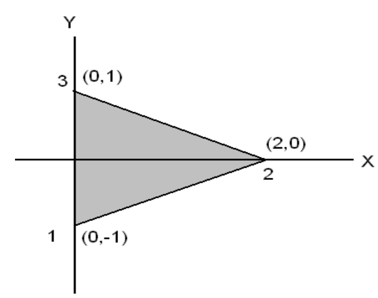
Temperature Effects

The thermal force vector is given by

$$\{f\}_t = 2\prod rA[B][D]\{e\}_t$$

$$\{f\}_t = \begin{Bmatrix} F_1u \\ F_1w \\ F_2u \\ F_2w \\ F_3u \\ F_3w \end{Bmatrix}$$

3. **Problem :-** Assuming plane stress conditions evaluate the stiffness matrix for the element shown in Fig. Assume E= 2 x 10^5 N/cm^2 and μ=0.3. u_1=0.000, v_1=0.0025, u_2=0.0012, v_2=0.000, u_3 =0.0000 & v_3= 0.0025.



$\beta_1 = y_2 - y_3 = 0 - 1 = -1$ $\qquad$ $\gamma_1 = -(x_2 - x_3) = 0 - 2 = -2$

$\beta_2 = y_3 - y_1 = 1 + 1 = 2$ $\qquad$ $\gamma_2 = -(x_3 - x_1) = 0 - 0 = 0$

$\beta_3 = y_1 - y2 = -1 - 0 = -1$ $\qquad$ $\gamma_3 = -(x_1 - x_2) = 2 - 0 = 2$

. A = ½ x b x h = ½ x 2 x 2 = 2

$$\{\in\} = \frac{1}{2A}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \{B\}\ \{d\}$$

$$[B] = \frac{1}{2(2)} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{pmatrix}$$

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

Now we know that the stiffness matrix [K] is given by $\int_{vol} [B]^T [D] [B] dv$
Assuming unit thickness ie t = 1 we get
$[K] = A\, [B]^T [D] \{B]$

$$= \frac{(2)\,(2 \times 10^5)}{4\,(0.91)} \underset{6 \times 3}{\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 1 & 0.35 \end{bmatrix}} \; \tfrac{1}{4} \underset{3 \times 6}{\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}}$$

$$= 1.099 \times 10^2 \underset{6 \times 6}{\begin{bmatrix} 600 & 325 & -500 & -350 & -100 & 25 \\ 325 & 1087.5 & -300 & -175 & -25 & -912.5 \\ -500 & -300 & 1000 & 0 & -500 & 300 \\ -350 & -175 & 0 & 350 & 350 & -175 \\ -100 & -25 & -500 & 350 & 600 & -325 \\ 25 & -912.5 & 300 & -175 & -325 & 1087.5 \end{bmatrix}}$$

Note: In order to evaluate the element stress we can use the equation
$\{\sigma\} = [D]\,[B]\,\{d\}$

4. Derivation of shape function for axisymmetric element. (triangular element)

Consider an axisymmetric triangular element with nodes 1, 2 and 3 shown in fig.
Let the nodal displacement be u_1 w_1, u_2 w_2, and u_3 w_3.

$$\text{Displacement, } \{u\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

Since the triangular element has two degree of freedom at each node, it has 6 generalized co-ordinates.

Displacement functions, $u = a_1 + a_2 r + a_3 z$

$W=a_4+a_5r+a_6z$

Where, a_1, a_2, a_3, a_4, a_5, a_6 are global or generalized co-ordinates.

Let $u_1= a_1+a_2r_1+a_3z_1$

$u_2= a_1+a_2r_2+a_3z_2$

$u_3= a_1+a_2r_3+a_3z_3$

write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{Let } D=\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}$$

$$D^{-1}=\frac{C^T}{|D|}$$

Find the co-factors of matrix D.

$C_{11}=+\begin{vmatrix} r_2 & z_2 \\ r_3 & z_3 \end{vmatrix}=(r_2z_3-r_3z_2)$

$C_{12}=-\begin{vmatrix} 1 & z_2 \\ 1 & z_3 \end{vmatrix}=-(z_3-z_2)=(z_2-z_3)$

$C_{13}=+\begin{vmatrix} 1 & r_2 \\ 1 & r_3 \end{vmatrix}=+(r_3-r_2)$

$C_{21}=\begin{vmatrix} r_1 & z_1 \\ r_3 & z_3 \end{vmatrix}=-(r_1z_3-r_3z_1)=r_3z_1-r_1z_3$

$C_{22}=+\begin{vmatrix} 1 & z_1 \\ 1 & z_3 \end{vmatrix}=(z_3-z_1)$

$C_{23}=-\begin{vmatrix} 1 & r_1 \\ 1 & r_2 \end{vmatrix}=-(r_3-r_1)=(r_1-r_3)$

$C_{31}=+\begin{vmatrix} r_1 & z_1 \\ r_2 & z_2 \end{vmatrix}=r_1z_2-r_2z_1$

$C_{32}=-\begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix}=-(z_2-z_1)=(z_1-z_2)$

$C_{33}=+\begin{vmatrix} 1 & r_1 \\ 1 & r_2 \end{vmatrix}=(r_2-r_1)$

$$C=\begin{vmatrix}(r_2z_3-r_3z_2) & (z_2-z_3) & (r_3-z_2)\\(r_3z_1-r_1z_3) & (z_3-z_1) & (r_1-r_3)\\(r_1z_2-r_2z_1) & (z_1-z_2) & (r_2-r_1)\end{vmatrix}$$

$$C^T=\begin{vmatrix}r_2z_3-r_3z_2 & z_2-z_3 & r_3-z_2\\r_3z_1-r_1z_3 & z_3-z_1 & r_1-r_3\\r_1z_2-r_2z_1 & z_1-z_2 & r_2-r_1\end{vmatrix}$$

We know that,

$$D=\begin{bmatrix}1 & r_1 & z_1\\1 & r_2 & z_2\\1 & r_3 & z_3\end{bmatrix} \qquad |D|=\begin{bmatrix}1 & r_1 & z_1\\1 & r_2 & z_2\\1 & r_3 & z_3\end{bmatrix}$$

$$|D|=1(r_2z_3-r_3z_2)\text{-r}_1(z_3-z_2)\text{+z}_1(r_3-r_2)$$

Substitute C^T and D values in equation

$$D^{-1}=\frac{1}{(r_2z_3-r_3z_2)-\text{r}_1(z_3-z_2)+\text{z}_1(r_3-r_2)}\ \text{x}\begin{vmatrix}r_2z_3-r_3z_2 & z_2-z_3 & r_3-z_2\\r_3z_1-r_1z_3 & z_3-z_1 & r_1-r_3\\r_1z_2-r_2z_1 & z_1-z_2 & r_2-r_1\end{vmatrix}$$

$$\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\begin{bmatrix}1 & r_1 & z_1\\1 & r_2 & z_2\\1 & r_3 & z_3\end{bmatrix}^{-1}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$$

$$=\frac{1}{(r_2z_3-r_3z_2)-\text{r}_1(z_3-z_2)+\text{z}_1(r_3-r_2)}\ \text{x}\begin{vmatrix}r_2z_3-r_3z_2 & z_2-z_3 & r_3-z_2\\r_3z_1-r_1z_3 & z_3-z_1 & r_1-r_3\\r_1z_2-r_2z_1 & z_1-z_2 & r_2-r_1\end{vmatrix}\text{x}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$$

Then area of the triangle can be expressed as a function of the r,z co-ordinates of the nodes 1,2 and 3.

$$A=\frac{1}{2}\begin{bmatrix}1 & r_1 & z_1\\1 & r_2 & z_2\\1 & r_3 & z_3\end{bmatrix}$$

$$A=\frac{1}{2}[1(r_2z_3-r_3z_2)\text{-r}_1(z_3-z_2)\text{+z}_1(r_3-r_2)]$$

$$2A=\left(r_2 z_3 - r_3 z_2\right)-r_1\left(z_3 - z_2\right)+z_1\left(r_3 - r_2\right)$$

Substituting equation,

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{vmatrix} r_2 z_3 - r_3 z_2 & z_2 - z_3 & r_3 - z_2 \\ r_3 z_1 - r_1 z_3 & z_3 - z_1 & r_1 - r_3 \\ r_1 z_2 - r_2 z_1 & z_1 - z_2 & r_2 - r_1 \end{vmatrix} \text{x} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \text{x} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\alpha_2 = r_3 z_1 - r_1 z_3$; $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_1$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$;

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$;

$u=a_1+a_2 r+a_3 z$

we can write this equation in matrix form,

$$u=\begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & r & z \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \text{x} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [\alpha_1 + \beta_1 r + \gamma_1 z \;\; \alpha_2 + \beta_2 r + \gamma_2 z \;\; \alpha_3 + \beta_3 r + \gamma_3 z] \text{x} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$U= \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A} \;\; \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A} \;\; \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A} \text{ x} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

U= $\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$ W= $\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$

Where, shape function, $N_1 = \dfrac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$ $N_2 = \dfrac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$ $N_3 = \dfrac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$

We can write equation as follow

$U = N_1u_1 + N_2u_2 + N_3u_3$

$W = N_1w_1 + N_2w_2 + N_3w_3$

Assembling the equation in matrix form,

$$u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

5. **Derive the strain-displacement matrix [B] for axisymmentric triangular element**

Displacement function for axisymmetric triangular element is given by,

$$\text{Displacement function, } u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

We can write, $u = N_1u_1 + N_2u_2 + N_3u_3$

$W = N_1w_1 + N_2w_2 + N_3w_3$

The strain components are,

Radial strain, $e_r = \dfrac{\partial u}{\partial r} = \dfrac{\partial N_1}{\partial r} u_1 + \dfrac{\partial N_2}{\partial r} u_2 + \dfrac{\partial N_3}{\partial r} u_3$

$$e_r=\frac{\partial N_1}{\partial r}u_1+\frac{\partial N_2}{\partial r}u_2+\frac{\partial N_3}{\partial r}u_3$$

circumferential strain, $e_\Theta=\frac{N_1}{r}u_1+\frac{N_2}{r}u_2+\frac{N_3}{r}u_3$

longitudinal strain, $e_z=\frac{\partial w}{\partial z}$

$$e_z=\frac{\partial N_1}{\partial r}w_1+\frac{\partial N_2}{\partial r}w_2+\frac{\partial N_3}{\partial r}w_3$$

shear strain, $\gamma_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}$

$$\gamma_{rz}=\frac{\partial N_1}{\partial r}u_1+\frac{\partial N_2}{\partial r}u_2+\frac{\partial N_3}{\partial r}u_3+\frac{\partial N_1}{\partial r}w_1+\frac{\partial N_2}{\partial r}w_2+\frac{\partial N_3}{\partial r}w_3$$

Arranging equation in matrix form

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}=\begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix}\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

From equation, we know that

$N_1=\frac{\alpha_1+\beta_1 r+\gamma_1 z}{2A}$ $N_2=\frac{\alpha_2+\beta_2 r+\gamma_2 z}{2A}$ $N_3=\frac{\alpha_3+\beta_3 r+\gamma_3 z}{2A}$

Partial differentiation- $\frac{\partial N_1}{\partial r}=\frac{\beta_1}{2A}$ $\frac{\partial N_2}{\partial r}=\frac{\beta_2}{2A}$ $\frac{\partial N_3}{\partial r}=\frac{\beta_3}{2A}$

$$\frac{N_1}{r}=\frac{1}{2A}\left(\frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r}\right) \quad \frac{N_2}{r}=\frac{1}{2A}\left(\frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r}\right)$$

$$\frac{N_3}{r}=\frac{1}{2A}\left(\frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r}\right)$$

$$\frac{\partial N_1}{\partial r} = \frac{\gamma_1}{2A} \qquad \frac{\partial N_2}{\partial r} = \frac{\gamma_2}{2A} \qquad \frac{\partial N_3}{\partial r} = \frac{\gamma_3}{2A}$$

Substitute the values.

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

The above equation n the form of

{e}=[B] {u}

Where, [B]=strain displacement metrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 \qquad \alpha_2 = r_3 z_1 - r_1 z_3 \; \alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3 \; \beta_2 = z_3 - z_1 \qquad \beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2 \; \gamma_2 = r_1 - r_3 \qquad \gamma_3 = r_2 - r_1$

6. **Derive the stress - strain matrix [D] for axisymmentric triangular element.**

By using Hooke’s law, we derived the following stresses equations,

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[e_x(1-v) + ve_y + ve_z\right]$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)}\left[ve_x(1-v) + ve_y + ve_z\right]$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[ve_x(1-v) + ve_y + ve_z\right]$$

$$\tau_{xz} = \frac{E}{(1+v)(1-2v)} X \left(\frac{1-2v}{2}\right) X \gamma_{xz}$$

Substitute x=r and y=Ɵ in the above equations,

Radial stress, $\sigma_r = \frac{E}{(1+v)(1-2v)}\left[e_r(1-v) + ve_\theta + ve_z\right]$

Circumferential stress, $\sigma_\theta = \frac{E}{(1+v)(1-2v)}\left[ve_r(1-v) + ve_\theta + ve_z\right]$

Longitudinal stress, $\sigma_z = \frac{E}{(1+v)(1-2v)}\left[ve_r + ve_\theta + (1-v)ve_z\right]$

Shear stress, $\tau_{xz} = \frac{E}{(1+v)(1-2v)} X \left(\frac{1-2v}{2}\right) X \gamma_{xz}$

Arranging the above equations, in matrix form

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

The above equation is in the form of,

{σ}=[D] {e}

Where, [D]=stress –strain relationship matrix,

$$= \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

7. Derive element stiffness matrix for axi-symmetric triangular element.

Element stiffness matrix [K] for Axi-symmetric triangular element We know that,

Stiffness matrix, [k] = $\int_v [B]^T [D][B] dv$

Where, v---volume=2ΠrA Coordinates,r = (r1+r2+r3)/3

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2$ $\alpha_2 = r_3 z_1 - r_1 z_3$ $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$ $\beta_2 = z_3 - z_1$ $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$ $\gamma_2 = r_1 - r_3$ $\gamma_3 = r_2 - r_1$

BODY FORCE
A body force is distributed force acting on every elemental volume of the body
Unit: Force per unit volume.
Example: Self weight due to gravity

$$2\pi\int u^T f_r dA = \begin{bmatrix} u1 \\ w1 \\ u2 \\ w2 \\ u3 \\ w3 \end{bmatrix} f$$

TEMPERATURE EFFECTS:

When the free expansion is prevented in the axisymmetric element, the change in temperature causes stresses in the element.

Let ΔT be the raise in temperature and α be the co-efficient of thermal expansion. The thermal force vector due to raise in temperature is given by

{F}t = [B]T [D] {e}t X 2ΠrA

For axi-symmetric triangular element

$$[F]t = \begin{bmatrix} F_{1u} \\ F_{1w} \\ F_{2u} \\ F_{2w} \\ F_{3u} \\ F_{3w} \end{bmatrix} \text{and strain } \{e\}t = \begin{bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \\ \alpha\Delta T \end{bmatrix}$$

STRESS CALCULATION

Elemental StressesIn Any Axi-Symmetric Problems Are As Follows

- Radial strain
- Circumferential strain
- Longitudinal strain
- Shear strain

Radial strain, $e_r = \frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3$

$$e_r = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3$$

circumferential strain, $e_\Theta = \frac{N_1}{r} u_1 + \frac{N_2}{r} u_2 + \frac{N_3}{r} u_3$

longitudinal strain, $e_z = \frac{\partial w}{\partial z}$ $\quad e_z = \frac{\partial N_1}{\partial r} w_1 + \frac{\partial N_2}{\partial r} w_2 + \frac{\partial N_3}{\partial r} w_3$

shear strain, $\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

8. **Problem: The nodal co-ordinates for an axisymmetric triangular element are given below:**

r_1 = 10 mm $\quad$ z_1 = 10 mm $\quad$ r_2 = 30 mm $\quad$ z_2 = 10 mm

r_3 = 30 mm $\quad$ z_3 = 40 mm $\quad$ Evaluate [B] matrix for the element

Given : Co-ordinates: r_1 = 10 mm, z_1 = 10 mm; r_2 = 30 mm, z_2 = 10 mm, r_3 = 30 mm, z_3 = 40 mm

To find: Displacement matrix [B].

Because of symmetry the stress components are independent of the angular co-ordinate 'θ' and hence all the derivatives with respect to 'θ' vanish and the components $\gamma_{x\theta}$, $\gamma_{r\theta}$are zero. The strain displacement relation are given by

$$\varepsilon_r = \frac{\partial u}{\partial r} \;;\; \varepsilon_\theta = \frac{u}{r} \;;\; \varepsilon_z = \frac{\partial w}{\partial z} \;;\; \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$N_i\ (x,y) = \frac{1}{2A_e}\ (\alpha_i + \beta_i r + \gamma_i z)$$

$$\alpha_i = r_j\, z_k - z_k\, r_j$$

$$\beta_i = z_j - z_k$$

$$\gamma_i = r_k - r_j$$

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} = Bd = \frac{1}{2A} \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_2}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

SOLUTION

Strain Displacement matrix

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Where, A = Area of triangular element

$$= \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{2}[(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2)]$$

$$= \frac{1}{2}[(30 \times 40) - (30 \times 10) - 10(40 - 10) + (30 - 30)]$$

$$= \frac{1}{2} \times (600)$$

$$\therefore\ A = 300\, mm^2$$

Co-ordinates:

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{10 + 30 + 30}{3}$$
$$r = 23.3334$$
$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 10 + 40}{3}$$
$$z = 20$$

$\alpha_1 = r_2z_2 - r_3z_2 = (30x40) - (30x10) = 900$

$\alpha_2 = r_3z_1 - r_1z_3 = (30x10) - (10x40) = -100$

$\alpha_3 = r_1z_2 - r_2z_1 = (10\ x\ 10) - (30x10) = -200$

$\beta_1 = z_2 - z_3 = 10\text{-}40 = -30$

$\beta_2 = z_3 - z_1 = 40\text{-}10 = 30$

$\beta_3 = z_1 - z_2 = 10\ \text{-}10 = 0$

$\gamma_1 = r_3 - r_1 = 30\text{-}30 = 0$

$\gamma_2 = r_3\ \text{-}r_1 = 10\text{-}30 = -20$

$\gamma_3 = r_2 - r_1 = 30\text{-}10 = 20$

$$\Rightarrow \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{900}{23.334} + (-30) + \frac{0 \times 20}{23.334} = 8.571$$

$$\Rightarrow \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{-100}{23.334} + (30) + \frac{(-20 \times 20)}{23.334} = 8.571$$

$$\Rightarrow \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-200}{23.334} + (0) + \frac{20 \times 20}{23.334} = 8.571$$

Substituting the values in strain displacement matrix

Stresses:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \underbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}}_{[D]} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix}$$

9. Differentiate plate and shell elements used in FEA

PLATE AND SHELL ELEMENTS

Plate is a flat surface having considerably large dimensions as compared to its thickness. Slabs in civil

engineering structures, bearing plates under columns, many parts of mechanical components are the common examples of plates. In this chapter, we are considering bending of such plates under lateral loads. The bending properties of a plate depend greatly on its thickness. Hence in classical theory we have the following groups

(i) thin plates with small deflections

(ii) thin plates with large deflections, and

(iii) thick plates

In thin plates with small deflections theory, the following assumption are made

(a) There is no deformation in the middle plane of the plate. This plane remains neutral during bending.

(b) Points of the plate lying initially on a normal to the middle surface of the plate remain on the normal to the same surface even after bending.

(c) The normal stresses in the direction transverse to the plate are negligible.

This theory is satisfactory for plates with ratio of thickness to span exceeding 1/10 and the ratio of maximum deflection to thickness less then 1/5

. Many engineering problems lie in the above category and satisfactory results are obtained by classical theories of thin plates.

Stresses in the middle plane are negligible, if the deflections are small in comparison with thickness. If the deflections are large, the in plane stresses developed in the so called neutral surfaces are to be considered. This gives rise to theory of thin plates with large deflections, in which geometric non-linearity is incorporated.

Displacement models for plate analysis

Category I: C2-Continuity element i.e. second order continuity elements in which secondderivates of ‘w’ are also nodal unknowns.
Category II: C1-Continuity elements i.e. first order continuity elements in which highest order of derivatives of ‘w’ is one only.
Category III: C0-Continuity element i.e. the elements in which only continuity of nodal variables are to be ensured.

Analysis of shell element:

A shell is a curved surface. Due to their shape they transfer most of the load applied on their surface as in plane forces (membrane forces) rather than by flexure. Hence the shells are examples of strength through form rather than mass.

Civil engineers use them as roofs to get large column free areas covered. Cylindrical shells, domes hyperbolic parabolic shells etc. are common examples of shell roofs.

Cooling towers, conical shells are also commonly used shells.

Mechanical and chemical engineers use shells as pressure vessels and as components of many machines.

Shells may be classified as singly curved or doubly curved. Classification of shell surfaces is attempted on the basis of Gauss curvature (product of principle curvature in two perpendicular directions).

If the Gauss Curvature is positive, zero, negative the surface will be classified as synclastic, developable, anticlastic respectively. Further classification is possible depending upon whether a shell is translational surface, a ruled surface or a surface of revolution.

Forces on Shell element
A typical shell element and various stress resultants acting on it. It may be noted that the sign convention is:

(i) Coordinate direction are as per right hand thumb rule
(ii) A force acting on +ve face in +ve direction or –ve face –ve direction is +ve
(iii) A +ve force acting on +ve z-direction produces +ve moment, about mid surface

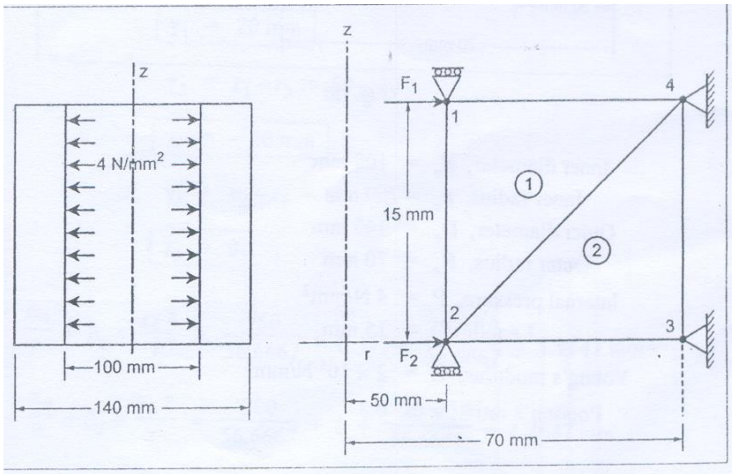
The four different approaches used to generate the shell are
1. Flat Elements
2. Curved Elements
3. Solid Elements
4. Degenerated Solid Elements.

Flat quadrilateral shell elements have use limitations even in linear analysis,since a mesh that consists of strictly flat elements may be impossibly to construct over a doubly-curved shell surface. For large deflection nonlinear analysis this deficiency becomes more pronounced. Even if the initial mesh satisfies the flat element restriction, the deformations can become so large that warping of the elements can be significant. Finding ways of handling warped element geometries is thus of fundamental importance for quadrilateral shell elements.

The current approach to deriving the quadrilateral plate bending element utilizes reference lines. Hrennikoff [00] first used this concept for plate modeling where the goal was to come up with a beam framework useful as a model for bending of flat plates.

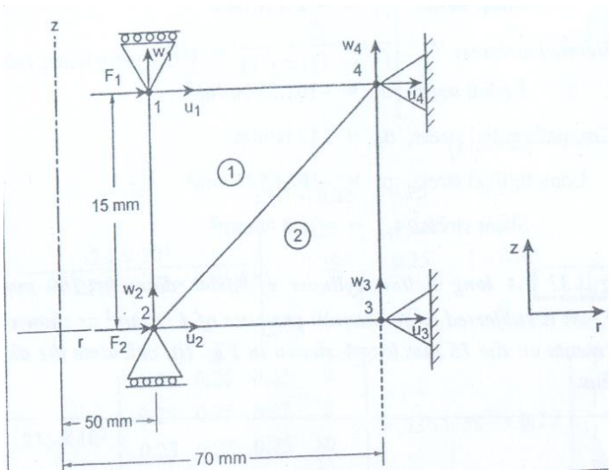
Park and Stanley [00, 00] used the reference line concept in their development of several plate and shell elements based on the ANS formulation. The reference lines were used to find beam-like curvatures; these curvatures were then used to find the plate curvatures through various Assumed Natural Strain distributions. These plate and shell elements were of Mindlin-Reissner type, and the reference lines were treated as Timoshenko beams. The present element is a Kirchhoff type plate and the reference lines are thus treated like Euler-Bernoulli (or Hermitian) beams.

10. **A hollow cylinder of inside diameter 100mm and outside diameter 140mm is subjected to an internal pressure of 4 N/mm^2. As shown in fig. by using two elements on the 15mm length shown in fig. calculate the displacements at the inner radius.
Take E=2x10^5 N/mm^2 and v=0.3.**



(16)

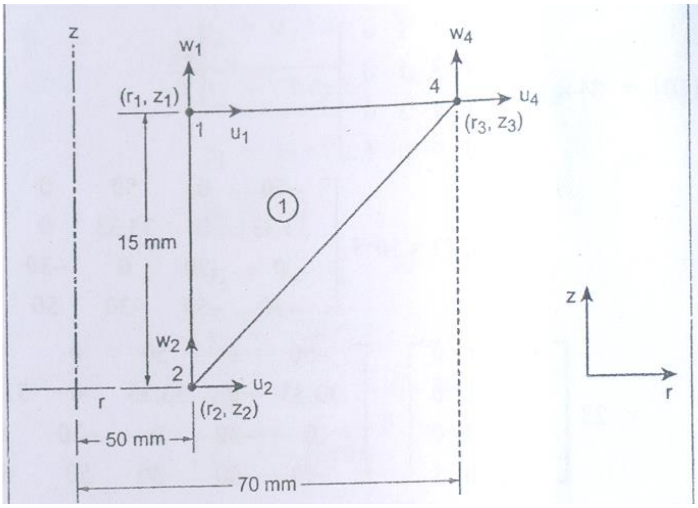
Given:



Inner diameter, d_e=100mm
Inner radius, r_e=50mm
Outer diameter, D_e=140mm
Outer radius, R_e=70mm
Internal pressure, p=4 N/mm^2
Young's modulus, E=2x10^5 N/mm^2
Poisson's ratio=0.3

To find: nodal displacement: $u_1,w_1,u_2,w_2,u_3,w_3,u_4,w_4$.

Solution:



For element (1) nodal displacements, $u_1,w_1,u_2,w_2,u_3,w_3,u_4,w_4$.

Co-ordinates:

At node 1:

r_1=50mm

z_1=15mm

At node 2:

r_2=50mm

z_2=0mm

At node 3:

r_3=70mm

z_3=15mm

we know that, r= $\frac{r_1 + r_2 + r_3}{3} = \frac{50+50+70}{3}$

r=56.66667mm

z= $\frac{z_1 + z_2 + z_3}{3} = \frac{15+0+15}{3}$

z=10mm

area of the triangle element=1/2xbreath x height=1/2x(70-50)x15

A=150mm

We know that matrix for axisymmetric triangular element (1)

$[K]_1 = 2\pi rA\ [B]^T\ [D]\ [B]$

Stress-strain relationship matrix, $[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$

$$[D]=\frac{2x10^5}{0.5}\begin{bmatrix} 1-0.3 & 0.3 & 0. & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.3 & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-2x0.3}{2} \end{bmatrix}$$

$$=\frac{2x10^5}{0.5}\begin{bmatrix} 1-0.3 & 0.3 & 0. & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.3 & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-2x0.3}{2} \end{bmatrix}$$

We know that,

Strain-displacement matrix,

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = (50X15)-(70X0)$

$\alpha_1 = 750mm^2$

$\alpha_2 = r_3 z_1 - r_1 z_3 = (70X15)(50X15)$

$\alpha_2 = 300mm^2$

$\alpha_3 = r_1 z_2 - r_2 z_1 = (50X0)-(50X15)$

$\alpha_3 = -750mm^2$

$\beta_1 = z_2 - z_3 = 0-15$

$\beta_1 = -15mm$

$\beta_2 = z_3 - z_1 = 15-15$

$\beta_2 = 0mm$

$\beta_3 = z_1 - z_2 = 15-0$

$\beta_3 = 15mm$

$\gamma_1 = r_3 - r_2 = 70-50$

$\gamma_1 = 20mm$

$$\gamma_2 = r_1 - r_3 = 50 - 70$$
$$\gamma_2 = -20mm$$
$$\gamma_3 = r_2 - r_1 = 50 - 50$$
$$\gamma_3 = 0mm$$
$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{750}{56.667} + (-15) + \frac{20X10}{56.6667} = 1.7647mm$$
$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{300}{56.667} + 0 + \frac{-20X10}{56.6667} = 1.7647mm$$
$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-750}{56.667} + 15 + 0 = 1.7647mm$$

Substitute these values.

$$[B] = \frac{1}{2X150}\begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B] = 3.333X10^{-3}\begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B][D] = 384.6153X10^{3}\begin{bmatrix} 0.7 & 0.3 & 0. & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & \begin{matrix} 0 \\ 0.2 \end{matrix} \end{bmatrix} X\,3.333X10^{-3}\begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B][D] = 1.282X10^{3}\begin{bmatrix} -9.9706 & 6 & .5294 & -6 & 11.0294 & 0 \\ -3.2647 & 6 & 1.2353 & -6 & 5.7353 & 0 \\ -3.9706 & 14 & 0.5294 & -14 & 5.0294 & 0 \\ 4 & -3 & -4 & 0 & 0 & 3 \end{bmatrix}$$

We know that,

$$[B] = 3.333X10^{-3}\begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T = 3.333X10^{-3}\begin{bmatrix} -15 & 1.7647 & 0 & 20 \\ 0 & 0 & 20 & -15 \\ 0 & 1.7467 & 0 & -20 \\ 0 & 0 & -20 & 0 \\ 15 & 1.7467 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T[B][D] = 1.282X10^3\begin{bmatrix} -9.9706 & 6 & .5294 & -6 & 11.0294 & 0 \\ -3.2647 & 6 & 1.2353 & -6 & 5.7353 & 0 \\ -3.9706 & 14 & 0.5294 & -14 & 5.0294 & 0 \\ 4 & -3 & -4 & 0 & 0 & 3 \end{bmatrix} 3.333X10^{-3}\begin{bmatrix} -15 & 1.7647 & 0 & 20 \\ 0 & 0 & 20 & -15 \\ 0 & 1.7467 & 0 & -20 \\ 0 & 0 & -20 & 0 \\ 15 & 1.7467 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T[B][D] = 4.27333\begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.558 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.211 & -60 \\ 79.412 & -280 & -10.58 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

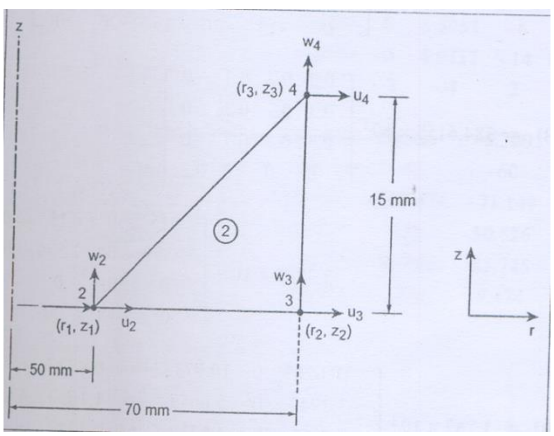
Substitute these values

$$[K]_1 = 2X\pi X56.667X150X4.27333\begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.558 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.211 & -60 \\ 79.412 & -280 & -10.58 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

$$[K]_1 = 228.2246X10^3X\begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.558 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.211 & -60 \\ 79.412 & -280 & -10.58 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

$$[K]_1 = 10^6X\begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & -35.448 & 13.693 \\ -31.817 & 74.173 & 16.10 & -63.903 & 22.957 & -10.270 \\ -19.573 & 16.110 & 18.755 & -2.46 & 2.310 & -13.693 \\ 18.124 & -63.903 & -2.416 & 63.903 & -22.957 & 0 \\ -35.448 & 22.957 & 2.310 & -22.957 & 40.068 & 0 \\ 13.693 & -10.270 & -13.693 & 0 & 0 & 10.270 \end{bmatrix}$$

For element:2



Co-ordinates:

At node 1:

r_1=50mm

z_1=0mm

At node 2:

r_2=70mm

z_2=0mm

At node 3:

r_3=70mm

z_3=15mm

we know that, r= $\frac{r_1 + r_2 + r_3}{3} = \frac{50 + 70 + 70}{3}$

r=63.3333mm

z= $\frac{z_1 + z_2 + z_3}{3} = \frac{0+0+15}{3}$

z=5mm

area of the triangle element=1/2xbreath x height=1/2x(70-50)x15

A=150mm

We know that matrix for axisymmetric triangular element (1)

$[K]_1=2\pi rA\ [B]^T\ [D]\ [B]$

Stress-strain relationship matrix, [D]= $\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$

$$[D]=\frac{2x10^5}{0.52}\begin{bmatrix} 1-0.3 & 0.3 & 0. & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.3 & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-2x0.3}{2} \end{bmatrix}$$

$$=\frac{2x10^5}{0.52}\begin{bmatrix} 1-0.3 & 0.3 & 0. & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.3 & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-2x0.3}{2} \end{bmatrix}$$

We know that,

Strain-displacement matrix,

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = (70X15)-(70X0)$

$\alpha_1 = 1050mm^2$

$\alpha_2 = r_3 z_1 - r_1 z_3 = (70X0)-(50X15)$

$\alpha_2 = -750mm^2$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (50X0) - (70X0)$$
$$\alpha_3 = 0mm^2$$
$$\beta_1 = z_2 - z_3 = 0 - 15$$
$$\beta_1 = -15mm$$
$$\beta_2 = z_3 - z_1 = 15 - 0$$
$$\beta_2 = 15mm$$
$$\beta_3 = z_1 - z_2 = 0 - 0$$
$$\beta_3 = 0mm$$
$$\gamma_1 = r_3 - r_2 = 70 - 70$$
$$\gamma_1 = 0mm$$
$$\gamma_2 = r_1 - r_3 = 50 - 70$$
$$\gamma_2 = -20mm$$
$$\gamma_3 = r_2 - r_1 = 70 - 50$$
$$\gamma_3 = 20mm$$
$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{1050}{63.333} + (-15) + 0 = 1.5647mm$$
$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{-750}{63.367} + 15 + \frac{-20X5}{63.6336} = 1.5647mm$$
$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{20X5}{63.33} = 1.579mm$$

Substitute these values.

$$[B] = \frac{1}{2X150}\begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B] = 3.333X10^{-3}\begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B][D] = 384.6153X10^3 \begin{bmatrix} 0.7 & 0.3 & 0. & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0 \\ & & & 0.2 \end{bmatrix} X\,3.333X10^{-3} \begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

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We know that,

$$[B] = 3.333X10^{-3} \begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T = 3.333X10^{-3} \begin{bmatrix} -15 & 1.7647 & 0 & 20 \\ 0 & 0 & 20 & -15 \\ 0 & 1.7467 & 0 & -20 \\ 0 & 0 & -20 & 0 \\ 15 & 1.7467 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T[B][D] = 1.282X10^3 \begin{bmatrix} -9.9706 & 6 & .5294 & -6 & 11.0294 & 0 \\ -3.2647 & 6 & 1.2353 & -6 & 5.7353 & 0 \\ -3.9706 & 14 & 0.5294 & -14 & 5.0294 & 0 \\ 4 & -3 & -4 & 0 & 0 & 3 \end{bmatrix} 3.333X10^{-3} \begin{bmatrix} -15 & 1.7647 & 0 & 20 \\ 0 & 0 & 20 & -15 \\ 0 & 1.7467 & 0 & -20 \\ 0 & 0 & -20 & 0 \\ 15 & 1.7467 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T[B][D] = 4.27333 \begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.558 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.211 & -60 \\ 79.412 & -280 & -10.58 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

Substitute these values

$$[K]_1 = 2X\pi X 56.667 X 150 X 4.27333 \begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.558 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.211 & -60 \\ 79.412 & -280 & -10.58 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

$$[K]_1 = 228.2246 X 10^3 X \begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.558 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.211 & -60 \\ 79.412 & -280 & -10.58 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

$$[K]_1 = 10^6 X \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & -35.448 & 13.693 \\ -31.817 & 74.173 & 16.10 & -63.903 & 22.957 & -10.270 \\ -19.573 & 16.110 & 18.755 & -2.46 & 2.310 & -13.693 \\ 18.124 & -63.903 & -2.416 & 63.903 & -22.957 & 0 \\ -35.448 & 22.957 & 2.310 & -22.957 & 40.068 & 0 \\ 13.693 & -10.270 & -13.693 & 0 & 0 & 10.270 \end{bmatrix}$$

Two Marks Question and Answers.

UNIT-III

1. **What is axi symmetric element?**

Many three dimensional problem in engineering exhibit symmetry about an axis of rotation such type of problem are solved by special two dimensional element called the axisymmetric element

2. **What are the conditions for a problem to be axisymmetric? (Apr 2010)**
   The condition to be axi-symmetric is as follows
   - The problem domain must be symmetric about the axis of revolution
   - All boundary condition must be symmetric about the axis of revolution
   - All loading condition must be symmetric about the axis of revolution

3. **Give the stiffness matrix equation for an axisymmetric triangular element.**

Stiffness matrix $[K]=[B]^T[D][B]2\pi rA$

A-Area r-Radius

4. **What is the purpose of Iso parametric element?**

It is difficult to represent the curved boundaries by straight edges finite elements. A large number of finite elements may be used to obtain reasonable resemblance between original body and the assemblage.

5. **Define super parametric element?**

If the number of nodes used for defining the geometry is more than of nodes used for defining the displacement is known as super parametric element.

6. **What is meant by Iso parametric element?**

If the number of nodes used for defining the geometry is same as number of nodes used for defining the displacement is known as Iso parametric element.

7. **Define sub parametric element?**

If the number of nodes used for defining the geometry is less than of nodes used for defining the displacement is known as sub parametric element

8. **Is beam elementan Iso parametric element?**

Beam element is not an Iso parametric element since the geometry and displacement are defined by different order interpretation functions.

9. **What is meant by degrees of freedom?**

When the force or reaction act at nodal point node is subjected to deformation.The deformation includes displacement rotation, and or strains. These are collectively known as degrees of freedom.

10. **State the principles of virtual energy?**

A body is in equilibrium if the internal virtual work equals the external virtual work for the every kinematically admissible displacement field.

11. **What is homogeneous form?**

When the specified values of dependent variables is zero, the boundary condition are said to be homogeneous.

12. **What is non-homogeneous form?**

When the specified values of dependent variables are non-zero, the boundary condition said to be non-homogeneous.

13. **Write down the shape functions for 4 noded rectangular elements using natural coordinate system?**

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \qquad N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

14. Write down the stress strain relationship matrix for plane stress conditions?

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$$

E-youngs modulus

v-poisson ratio

15. **Write down Jacobian matrix for 4noded quadrilateral elements?**

$$[J]=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

16. What is Aspect ratio?

It is defined as the ratio of the largest dimension of the element to the smallest dimension. In many cases, as the aspect ratio increases the in accuracy of the solution increases. The conclusion of many researches is that the aspect ratio should be close as unity as possible.

17. What is higher order element ? (Nov 2011)

When the finite element approximation is gradually refining the mesh in comparing the solution then it is called higher order elements

18. What are the four basic sets of elasticity equations? (May 2012)

Elasticity equations are used for solving structural mechanics problems. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. There are four basic sets of elasticity equations. They are:

- Strain- Displacement relationship equations
- Stress-Strain relationship equations
- Equilibrium equations
- Compatibility equations

19. What are the types of Non-Linearity ?(May 2012)

- Non-Linearity in material behavior from point to point
- Non-Linearity in loading deformation relation
- Geometric non-Linearity
- Change in boundary condition for different loading

20. What are higher order elements and why are they preferred? (April 2011)

- For any element, if the interpolation polynomial is the order of two or more, that element is known as higher order elements
- It is used to represent the curved boundaries
- The number of elements are reduced when compared with straight edge elements to model geometry

21. **Give four applications where axi-symmetric elements can be used? (April 2011)**

- Pressure Vessels
- Rocket Castings
- Cooling Towers
- Sub-Marine hulls
- Belleville springs

22. What is meant by plane stress analysis?

Plane stress is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.plane stress if the stress vector is zero across a particular surface. When that situation occurs over an entire element of a structure, as is often the case for thin plates, the stress analysis is considerably simplified, as the stress state can be represented by a tensor of dimension 2

23. Define plane strain analysis.(AU-NOV/DEC-2012)

Plane strain is defined to be a state of strain in which the strain normal to the xy plane and the shear strains are assumed to be zero.Plane strain is applicable to rolling, drawingandforging where flow in a particular direction is constrained by the geometry of the machinery, e.g. a well-lubricated die wall.

24. Write down the stress-strain relationship matrix for plane stress conditions.

For plane stain problem, stress strain relationship matrix is,

$$[D] = \frac{E}{1 - v_2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & v & \frac{1-v}{2} \end{bmatrix}$$

Where E=young’s modulus

V=Poisson’s ratio.

25. Write down the stress-strain relationship matrix for plane strain condition.

For plane strain problems, stress –strain relationship matrix is,

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \left(\frac{1-2v}{2}\right) \end{bmatrix}$$

3 noded triangular elements – four noded rectangular elements – higher order elements

26. What is axisymmetric element?(AU-APR/MAY-2014)

Many three dimensional problems in engineering exhibit symmetry about an axis of rotation. Such types of problems are solved by a special two dimensional element called as axisymmetric element. The axisymmetric problemor elements which is formulated from the three dimensional element into two dimensional element because of the symmetric axis. The symmetric axis which explains the side a is equal to the side a' which refers us to considerthe three dimensional problem into two dimensional one.

27. What are the conditions for a problem to be axisymmetric?

Conditions for a problem to be axisymmetric:

1.The problem domain must be symmetric about the axis of revolution.

2. All boundary conditions must be symmetric about the axis of revolution.determination of boundary condition is important in the axisymmetric probles.

3. All loading conditions must be symmetric about the axis of revolution.

28. Write down the displacement equation for an axisymmetric triangular element.

Displacement function, u(r,z)=j $\begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$

r,z represents the direction

u – represents displacement function in x axis

w-represents displacement function in y direction.

29. Write down the shape function for an axisymmetric triangular element. (AU-APR/MAY-2013)

Shape function, $N_1 = \dfrac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$

$N_2 = \dfrac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$

$N_3 = \dfrac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$

Where, $\alpha_1 = r_2 z_3 - r_3 z_2$

$\alpha_2 = r_3 z_1 - r_1 z_3$

$\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$

$\beta_2 = z_3 - z_1$

$\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$

$\gamma_2 = r_1 - r_3$

$\gamma_3 = r_2 - r_1$

30. Give the strain-displacement matrix equation for an axisymmetric triangular element.

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Where, co-ordinate, r= $\frac{r_1 + r_2 + r_3}{3}$

Co-ordinate, z= $\frac{z_1 + z_2 + z_3}{3}$

Where, $\alpha_1, \alpha_2\ \alpha_3, \gamma_1, \gamma_2\ \gamma_3, \beta_1, \beta_2, \beta_3$-co-ordinates

31. What are the types of non-linearity?

Types of non-linearity:

(i) Non linearity in material behavior from point to point.

(ii) Non-linearity loading-deformation reaction.

(iii) Geometric non-linearity.

(iv) Change in boundary condition for different loading.

natural coordinates and coordinate transformations – triangular and quadrilateral elements

32. Write down the stress-strain relationship matrix for an axisymmetric for an axisymmetric triangular element.(AU-APR/MAY-2012)

Stress-strain relationship matrix, $[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$

Where, E,-Young;s modulus

v-poisson’s ratio.

33. Give stiffness matrix equation for an axisymmetric triangular element

Stiffness matrix,

$[K] = 2\Pi r A [B] T [D][B]$

Where, co-ordinate

r= $\frac{r_1 + r_2 + r_3}{3}$

A-area of the triangular element matrix.

34. What are the ways in which a three dimensional problem can be reduced to a twodimension approach.

Reducing three dimensional element into two dimensional element:

(i) Plane stress; one dimensional is too small when compared to other two ic
(ii) dimensions.
(iii) Plane strain: one dimensional is too large when compared to other two dimensions
(iv) Axisymmetric: geometry about the axis.

35. What are natural co-ordinates? (AU-NOV/DEC-2013)

Natural coordinates:

A simple natural coordinate system is a kind of local coordinate system that permits the specification of a point within the element by a set of dimensionless numbers whose magnitude varies from -1 to +1. The actual finite element say two dimensional element which may be a rectangle or quadrilateral or any curved sided and specified any global coordinates system is represented in natural coordinate system.

36. What arethe advantagesofnaturalco-ordinates?

Advantages of natural coordinates:

i. Natural coordinate system enable us to formulate shape function easily.
ii. The shape function for the complicated structure also very easy in the natural coordinate system.
iii. The natural coordinate system helps us to solve the rectangle, quadrilateral, curve sided element in a simple way.

37. What is meant by axisymmetric solid? (AU-NOV/DEC-2012)

Axisymmetric solid:

In some three dimensional solid like cylinder, flywheel, turbine disc the material content is symmetric with respect to their axes. The material content is equal in equal distance of opposite sides. Hence the stress developed, displacement produced are considered as symmetric. Such solids are known as axisymmetric solids.

Due to this axisymmetric condition these three dimensional solids can be treated as two dimensional solids for analysis.

38. Specify the machine component related with axisymmetric concept. (AU-NOV/DEC-2011)

Machine component related with axisymmetric concept:

Pressure vessels- the stress deformation due to pressure are analysed here.
Pressure cylinders- the cylinders which carry internal or external loads.
Flywheel – energy storing element which may deformed during torque transmission.
Turbine discs- the rotary disc either pressure due to angle or force.
Soil mass – structural problem defines base plan of constructions.

39. Write short note on axisymmetric formulation.

Axisymmetric formulation:

$$\{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \left[\dfrac{\partial u}{\partial r} \quad \dfrac{u}{r} \quad \dfrac{\partial w}{\partial z} \left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \right) \right]^T$$

40. Brieflydiscussaboutfiniteelementmodelingforaxisymmetricformulation.

Finite element modeling for axisymmetric formulation:

For an axisymmetric element the two dimensional region defined by the revolving area can be divided into triangular elements or quadrilateral elements.

{F} = [K] {u}

$$[k] = \int_v [B]^T [D][B] dv$$

$[D] - stress - strain$ matrix

[B] - strain displacement matrix

[k] - stiffness matrix

{F} – Force

{u} - displacement

41. What are thenon zero strainandstress componentsofaxisymmetricelement?

Non zero strainandstress componentsofaxisymmetricelement:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

St.Anne's College of Engineering & Technology,
Department of Mechanical Engineering

Subject Name	:	**FINITE ELEMENT ANALYSIS**
Subject code	:	**ME8692**
Year	:	**IIIrd year**
Semester	:	**VIth semester**

UNIT V
ISOPARAMETRIC FORMULATION

Natural co-ordinate systems – Isoparametric elements – Shape functions for iso parametric elements – One and two dimensions – Serendipity elements – Numerical integration and application to plane stress problems - Matrix solution techniques – Solutions Techniques to Dynamic problems – Introduction to Analysis Software.

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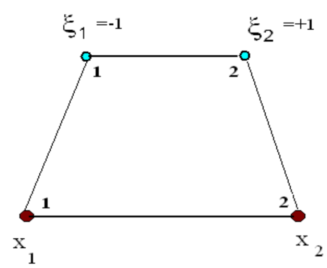
1. **Explain about the natural coordinate system used in FEA.**

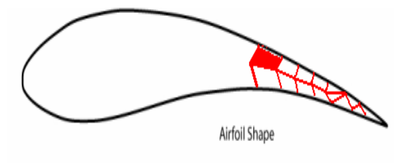
NATURAL CO-ORDINATE SYSTEMS

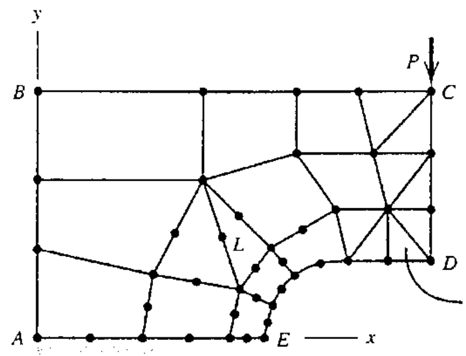
A Natural Co-ordinate system is a local co-ordinate system that permits the specification of a point within an element by a set of dimensionless numbers whose absolute magnitude never exceeds unity
i.e. A I Dimensional element described by means of its two end vertices (x_1& x_2) in Cartesian space is represented or mapped on to Natural co-ordinate space by the line whose end vertices ξ_1&ξ_2 are given by –1 & +1 respectively.

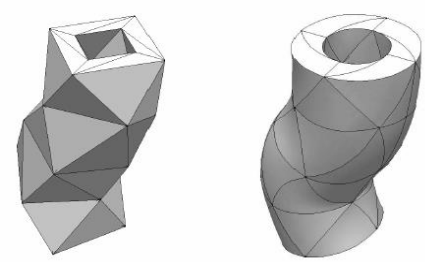


ADVANTAGES OF NATURAL CO-ORDINATE SYSTEMS

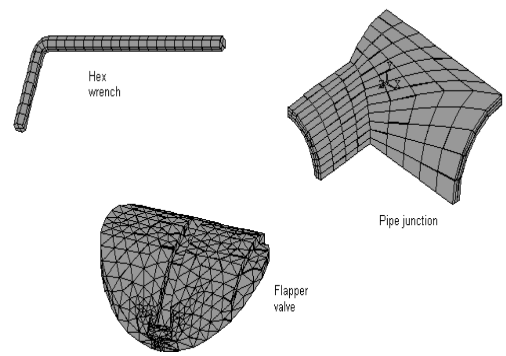
i. It is very convenient in constructing interpolation functions.
ii. Integration involving Natural co-ordinate can be easily performed as the limits of the Integration is always from –1 to +1. This is in contrast to global co-ordinates where the limits of Integration may vary with the length of the element.
iii. The nodal values of the co-ordinates are convenient number or fractions.
iv. It is possible to have elements with curved sides.

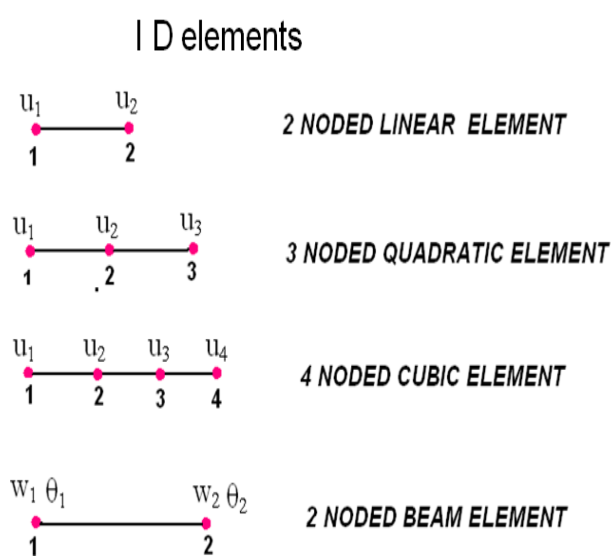


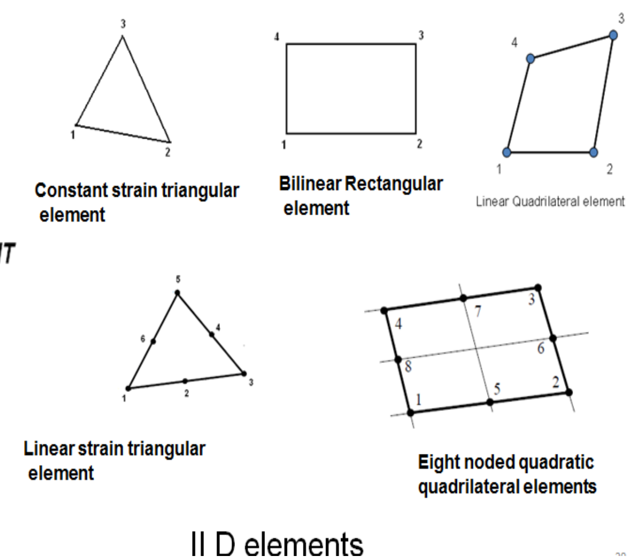


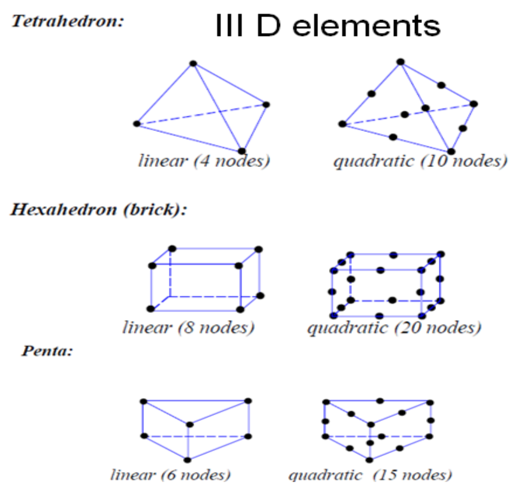


Straight-sided mesh (left) and curved mesh (right) for a 3D curved domain



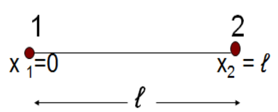






Linear Element:

We had derived the shape functions for a two noded linear element using Lagrangian polynomials.



$$L_1(x) = \frac{x - x_2}{x_1 - x_2}$$

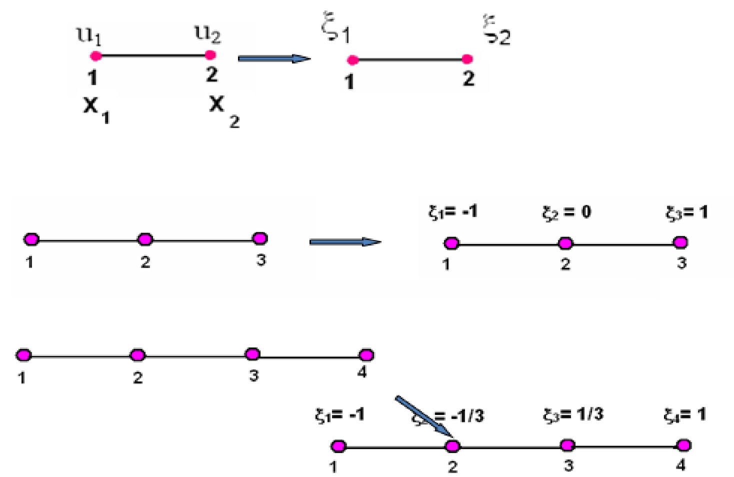
Substituting $x_1 = 0$ & $x_2 = \ell$ we get

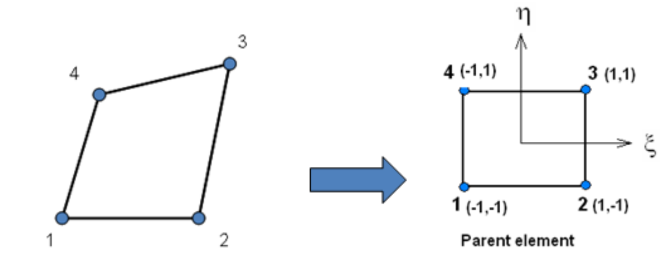
$$L_1(x) = \frac{x - \ell}{0 - \ell} = 1 - \frac{x}{\ell}$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1} = \frac{x}{\ell}$$

which are the same as that obtained by inverting the generalized co-efficient matrix.

Conversion into natural coordinates



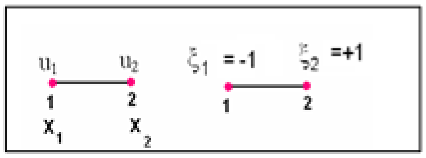


Linear Quadrilateral element

I - D Lagrangian Interpolation functions in Natural Co-ordinates

Linear Element:

$$L_1 = \frac{(\xi - \xi_2)}{(\xi_1 - \xi_2)}$$



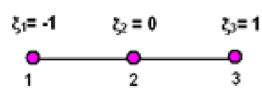
Substituting $\xi_1 = -1$ & $\xi_2 = +1$, we get

$$L_1 = \frac{(\xi - 1)}{-1 - 1} = \frac{1 - \xi}{2} = \frac{1}{2}(1 - \xi)$$

$$L_2 = \frac{(\xi - \xi_1)}{(\xi_2 - \xi_1)} = \frac{(\xi + 1)}{+1 + 1} = \frac{1}{2}(1 + \xi)$$

3 Noded Quadratic Element

$\xi_1 = -1 \quad \xi_2 = 0 \quad \xi_3 = 1$



$$L_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)\ (-1 - 1)} = \xi/2\,(\xi - 1)$$

$$= -\,\xi/2\,(1 - \xi)$$

$$L_2 = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)\ (\xi_2 - \xi_3)} = \frac{(\xi + 1)(\xi - 1)}{(0 + 1)\ \ (0 - 1)} = (1 - \xi)\,(1 + \xi)$$

$$L_3 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)\ (\xi_3 - \xi_2)} = \frac{(\xi + 1)(\xi - 0)}{(1 + 1)\ \ (1 - 0)} = \xi/2\,(1 + \xi)$$

4 Noded Cubic Element:

$$\xi_1 = -1 \quad \xi_2 = ⅓ \quad \xi_3 = ⅓ \quad \xi_4 = 1$$

$$L1 = \frac{(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_1 - \xi_4)}$$

$$= \frac{(\xi + ⅓)(\xi - ⅓)(\xi - 1)}{(-1 + ⅓)(-1 - ⅓)(-1 - 1)} = -\frac{9}{16}\left(\frac{1}{3} + \xi\right)(1 - \xi)\left(\frac{1}{3} - \xi\right)$$

$$L2 = \frac{(\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_1)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)(\xi_2 - \xi_4)}$$

$$= -\frac{27}{16}(1 + \xi)(1 - \xi)\left(\frac{1}{3} - \xi\right)$$

$$L_3 = \frac{(\xi - \xi_1)(\xi_1 - \xi_2)(\xi - \xi_4)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)(\xi_3 - \xi_4)}$$

$$= \frac{27}{16}(1 + \xi)(1 - \xi)\left(\frac{1}{3} + \xi\right)$$

$$L_4 = \frac{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)}{(\xi_4 - \xi_1)(\xi_4 - \xi_2)(\xi_4 - \xi_3)}$$

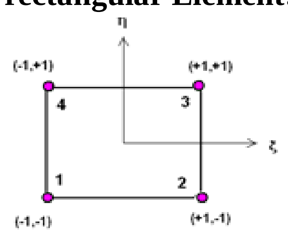
$$= -\frac{9}{16}\left(\frac{1}{3} + \xi\right)\left(\frac{1}{3} - \xi\right)(1 + \xi)$$

2. **Derive Lagrangian Interpolation polynomials for rectangular Element: (Natural Co-ordinates)**
 Lagrangian Interpolation polynomials for rectangular Element: (Natural Co-ordinates)

$$N_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1 - 1} = \frac{1 - \xi}{2}$$

$$N_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4} = \frac{\eta - 1}{-1 - 1} = \frac{1 - \eta}{2}$$

$$\therefore N_1(\xi, \eta) = - N_1(\xi)\, N_1(\eta)$$



$$= \left(\frac{1 - \xi}{2}\right)\left(\frac{1 - \eta}{2}\right)$$

$$= 1/4(1 - \xi)(1 - \eta)$$

$$N_2(\xi,\eta) = \frac{(\xi - \xi_1)(\eta - \eta_3)}{(\xi_2 - \xi_1)(\eta_2 - \eta_3)}$$

$$= \frac{(\xi + 1)(\eta - 1)}{(1 + 1)(-1 - 1)} \quad = ¼(1 + \xi)(1 - \eta)$$

$$N_3(\xi,\eta) = \frac{(\xi - \xi_4)(\eta - \eta_2)}{(\xi_3 - \xi_4)(\eta_3 - \eta_2)}$$

$$= \frac{(\xi + 1)(\eta + 1)}{(1 + 1)(1 + 1)} \quad = ¼(1 + \xi)(1 + \eta)$$

$$N_4(\xi,\eta) = \frac{(\xi - \xi_3)(\eta - \eta_1)}{(\xi_4 - \xi_3)(\eta_4 - \eta_1)}$$

$$= \frac{(\xi - 1)(\eta + 1)}{(-1 - 1)(1 + 1)}$$

$$= ¼(1 - \xi)(1 + \eta)$$

$$N_1(\xi,\eta) = ¼(1 - \xi)(1 - \eta)$$

$$N_2(\xi,\eta) = ¼(1 + \xi)(1 - \eta)$$

$$N_3(\xi,\eta) = ¼(1 + \xi)(1 + \eta)$$

$$N_4(\xi,\eta) = ¼(1 - \xi)(1 + \eta)$$

Bi-Linear rectangular Element:

$N_1(\xi,\eta) = - N_1(\xi)\, N_1(\eta)$ $= 1/4(1 - \xi)(1 - \eta)$

$N_2(\xi,\eta) = (\xi - \xi_1)(\eta - \eta_3)/(\xi_2 - \xi_1)(\eta_2 - \eta_3)$

$= (\xi + 1)(\eta - 1)/(1 + 1)(-1 - 1)$ $= ¼(1 + \xi)(1 - \eta)$

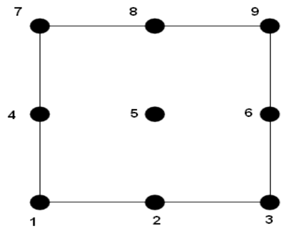
$N_3(\xi,\eta) = (\xi - \xi_4)(\eta - \eta_2) / (\xi_3 - \xi_4)(\eta_3 - \eta_2)$

$= (\xi + 1)(\eta + 1)/(1 + 1)(1 + 1)$ $= ¼(1 + \xi)(1 + \eta)$

$N_4(\xi,\eta) = (\xi - \xi_3)(\eta - \eta_1)/(\xi_4 - \xi_3)(\eta_4 - \eta_1)$

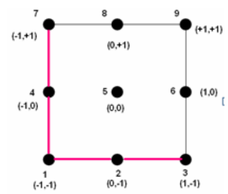
$= (\xi - 1)(\eta + 1)/(-1 - 1)(1 + 1)$ $= ¼(1 - \xi)(1 + \eta)$

3. **Derive shape function for Nine Noded Quadratic Quadrilateral element Nine Noded Quadratic Quadrilateral element**



$$N_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}$$

$$= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)}{2}$$

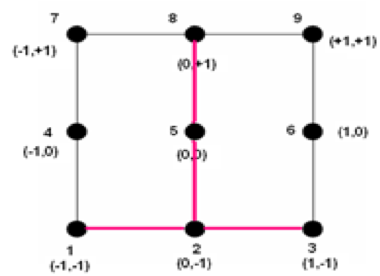


$$N_1(\eta) = \frac{(\eta - \eta_4)(\eta - \eta_7)}{(\eta_1 - \eta_4)(\eta_1 - \eta_7)}$$

$$= \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{\eta(\eta - 1)}{2}$$

$$\therefore N_1(\xi,\eta) = N_1(\xi)\, N_2(\eta) = \tfrac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$$

$N_1(\xi,\eta) = N_1(\xi)\, N_2(\eta) = \frac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$



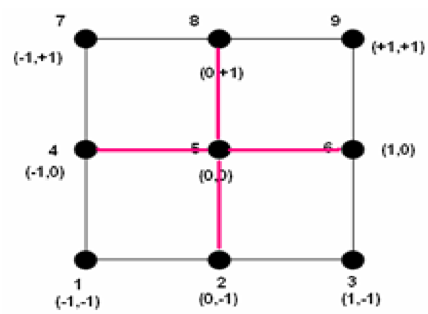
$N_2(\xi,\eta) = \frac{1}{2}(1 - \xi^2)(\eta^2-\eta)$

$N_3(\xi,\eta) = \frac{1}{4}(\xi^2 + \xi)(\eta^2-\eta)$

$N_4(\xi,\eta) = \frac{1}{2}(\xi^2 - \xi)(1 - \eta^2)$

$N_5(\xi,\eta) = (1 - \xi^2)(1 - \eta^2)$

$N_6(\xi,\eta) = \frac{1}{2}(\xi^2 + \xi)(1 - \eta^2)$



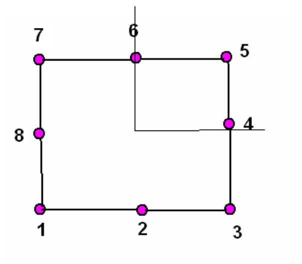
$N_7(\xi,\eta) = \frac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta)$

$N_8(\xi,\eta) = \frac{1}{2}(1 - \xi^2)(\eta^2 + \eta)$

$N_9(\xi,\eta) = \frac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta)$

4. **Derive Shape functions for Eight noded quadrilateral element :**

Shape functions for Eight noded quadrilateral element :



The equations to the various lines connecting the various nodes is given by

Line 1 – 2 – 3 ➞ $1 + \eta = 0$

Line 7 – 6 – 5 ➞ $1 - \eta = 0$

Line 1 – 8 – 7 ➞ $1 + \xi = 0$

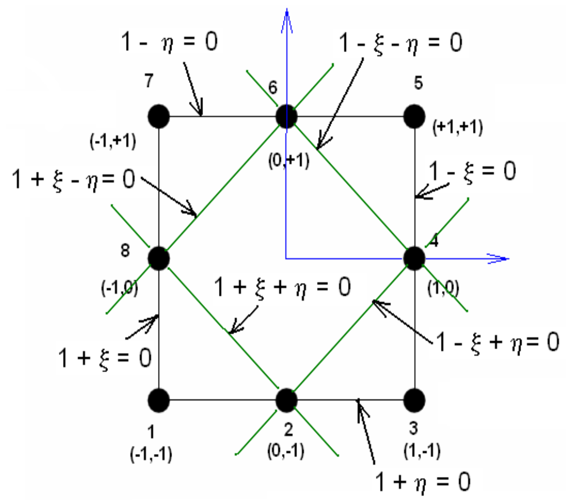
Line 3 – 4 – 5 ➞ $1 - \xi = 0$

Line 2 – 4 ➞ $1 - \xi + \eta = 0$

Line 6 – 8 ➞ $1 + \xi - \eta = 0$

Line 4 – 6 ➞ $1 - \xi - \eta = 0$

Line 8 – 2 ➞ $1 + \xi + \eta = 0$



To obtain the shape function N_1, we identify the equation to those lines not passing through node 1 and express N_1 as a product of these line equations.

i.e. lines 5-6-7, 3-4- 5 and 2-8

$\therefore N_1 = C(1 - \eta)(1 - \xi)(1 + \xi + \eta)$ $\therefore N_1(-1, -1) = C(1 + 1)(1 + 1)(1 - 1 - 1) = 1$ $\therefore C = -1/4$

$\therefore N_1(\xi,\eta) = -\frac{1}{4}(1 - \eta)(1 - \xi)(1 + \xi + \eta)$

Similarly for N_2 the lines are 1-8-7 , 3-4-5 and 5-6-7

$\therefore N_2 = C(1 - \eta)(1 + \xi)(1 - \xi) \quad = C(1 - \eta)(1 - \xi^2)$

$N_2(0, -1) = C(1 - 0)(1 + 1) = 1 \quad \therefore C = \frac{1}{2}$

$\therefore N_2(\xi,\eta) = \frac{1}{2}(1 - \xi^2)(1 - \eta)$

On similar lines we get the other shape functions: N_3, N_4, N_5, N_6, N_7 and N_8

$N_1(\xi,\eta) = -\frac{1}{4}(1 - \eta)(1 - \xi)(1 + \xi + \eta)$

$N_2(\xi,\eta) = \frac{1}{2}(1 - \xi^2)(1 - \eta)$

$N_3\ (\xi,\eta) = ¼\ (1 + \xi)\ (1 - \eta)\ (-1 + \xi - \eta)$

$N_4(\xi,\eta) = ½\ (1 - \xi)\ (1 - \eta^2)$

$N_5\ (\xi,\eta) = ½\ (1 + \xi)\ (1 - \eta^2)$

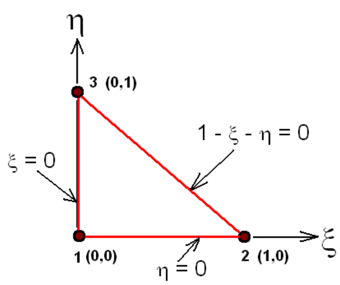
$N_6\ (\xi,\eta) = ¼\ (1 - \xi)\ (1 + \eta)\ (-1 - \xi + \eta)$

$N_7\ (\xi,\eta) = ½\ (1 - \xi^2)\ (1 + \eta)$

$N_8\ (\xi,\eta) = ¼\ (1 + \xi)\ (1 + \eta)\ (-1 + \xi + \eta)$

5. **Derive Shape functions for CST and LST element**

Shape functions for CST element



$N_1(\xi,\eta) = c\ (1 - \xi - \eta)$ $N_2(\xi,\eta) = c\ (\xi)$

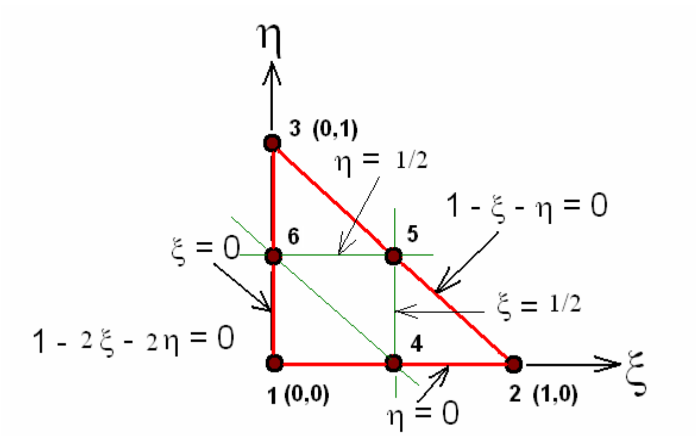
$N_1(0,0) = c\ (1)=1$ $N_2(1,0) = c\ (1) =1$

$\therefore c=1$ $\therefore c=1$

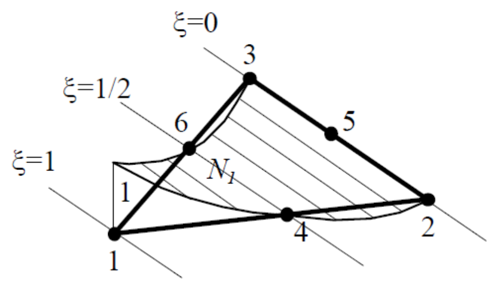
$\mathbf{N_1(\xi,\eta) = (1 - \xi - \eta)}$ $\mathbf{N_2(\xi,\eta) = \xi}$

Similarly $N_3= \eta$

Shape functions for LST element



$$N_1 = (1-\xi-\eta)(1-2\xi-2\eta)$$
$$N_2 = 2(\xi)(\xi-1/2)$$
$$N_3 = 2(\eta-1/2)(\eta)$$
$$N_4 = 4(\varepsilon)\ (1-\xi-\eta)$$
$$N_5 = 4\xi\eta$$
$$N_6 = 4(1-\xi-\eta)\eta$$



Shape Function N_1 for LST

6. What are isoparametric elements and explain its types.

ISOPARAMETRIC ELEMENTS

The element is said to be isoparametric when the shape functions defining the displacement pattern and geometry are the same and are of the same order.

Isoparametric elements are versatile and are used in two and three dimensional elasticity problems.

$$x = \sum_{i=1}^{r} x_i L_i(\xi)$$

For a linear transformation r = 2

$$\therefore\ x = x_1 N_1(\xi) + x_2 N_2(\xi)$$
$$= x_1 \frac{(1-\xi)}{2} + x_2 \frac{(1+\xi)}{2}$$

For example an element whose x co-ordinates are given by $x_1 = 3$ & $x_2 = 7$

Then $x = x_1 \frac{(1-\xi)}{2} + x_2 \frac{(1+\xi)}{2}$

$$3 = 3\frac{(1-\xi)}{2} + 7\frac{(1+\xi)}{2}$$

or $6 = 3 - 3\xi + 7 + 7\xi$ or $4\xi = -4$ or $\xi = -1$

ie the point $x_i = 3$ transforms to $\xi = -1$ in natural co-ordinate space

similarly $x_2 = x_1 \frac{(1-\xi)}{2} + x_2 \frac{(1+\xi)}{2}$

$$7 = 3\frac{(1-\xi)}{2} + 7\frac{(1+\xi)}{2}$$

$14 = 3 - 3\xi + 7 + 7\xi$

$4\xi = 4$ or $\xi = 1$

$\therefore$ The point $x_2 = 7$ in Cartesian space gets transformed to $\xi_2 = +1$ in Natural co-ordinate space. So the transformation

$$X = \sum_{i=1.}^{r} N_i\, x_i \ (\xi) \text{ transforms the geometry}$$

Similarly we have the approximation of the field variable in terms of shape functions expressed as

$$u = \sum_{i=1}^{s} u_i\, N_i\, (\xi)$$

Here 'r' - the number of nodes used for geometric transformation

's' - the number of nodes used for approximation of field variable.

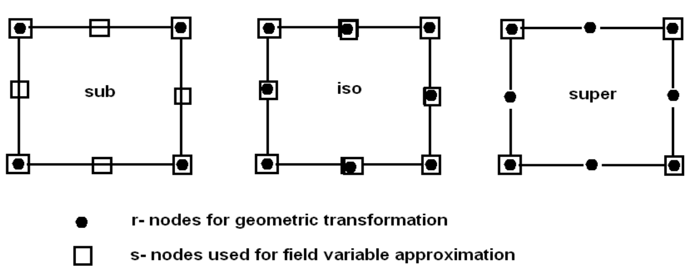
In general the polynomial used for geometric transformation need not be of the same order as that used for the field variable approximation. In other words two sets of nodes exists for the same region or element.

In other words two sets of nodes exists for the same region or element.

- One set of nodes for co-ordinate transformation from Cartesian space to natural co-ordinate space
- One set of nodes for approximating the variation of the field variable over the element.

Depending upon the relationship between these two polynomials elements are classified into three categories as

I. sub parametric elements $r < s$
II. iso-paramatric elements $r = s$
III. super-parametric elements $r > s$



Jacobian of Transformation

Among the 3 cases given above Isoparametric are more commonly used due to their advantages which include the following:

i) Quadrilateral elements in (x,y) coordinates with curved boundaries get transformed to a rectangle of (2 x 2) units in (ξ, η) co-ordinates

ii) Numerical integration is more easily performed as limits of integration vary from –1 to +1 for all elements.

We have seen that determination of the stiffness matrix requires the computation of derivative of shape functions with respect to 'x'. However as the shape functions (Interpolation function) are expressed in terms of ξ&η co-ordinates (natural co-ordinates) we use the chain rule.

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi}\frac{d\xi}{dx} = \frac{dN_1}{d\xi}\frac{1}{dx / d\xi}$$

$$= \frac{dN_1}{d\xi}\frac{1}{J} = J^{-1}\frac{dN_1}{d\xi}$$

Here J = dx/ dξ is the 'Jacobian' of transformations from Cartesian space to natural co-ordinate space. It can be considered as the scale factor between the two co-ordinate systems.

Jacobian of transformation for 2 Noded Linear Element

For a 2 Noded element the shape functions are given by

$N_1(\xi) = (1 - \xi)/2$

$N_2(\xi) = (1+\xi)/2$

Now $x = N_1 x_1 + N_2 x_2$

$= (1 - \xi)/2 \; x_1 + (1 +\xi) \; x_2$

$dx/d\xi = J = (-1 \; x_1)2+ \; x_2/2$

$= (x_2 – x_1)/2 = L/2$

Here (x_2 – x_1) represents the length of the element. So the Jacobian of transformation for a 2 noded element is given by L/2

3- Noded Quadratic element:-

$N_1 = -\xi/2\,(1-\xi)$
$N_2 = (1-\xi)\,(1+\xi)$
$N_3 = \xi/2\,(1+\xi)$
$u = N_1 u_1 + N_2 u_2 + N_3 u_3$ &
$x = N_1 x_1\ N_2 x_2 + N_3 x_3$

$$J = \frac{dx}{d\xi} = \begin{pmatrix} \frac{-1+2\xi}{2} & -2\xi & \frac{1+2\xi}{2} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Stiffness Matrix for a 2 Noded Axial Element

$$[K] = \int_0 B^T D\, BAdx$$

$$[B] = \frac{du}{dx} = \frac{dN}{dx} = \frac{1}{J}\frac{dN}{d\xi}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} \end{pmatrix}$$
$$= \frac{2}{L}\begin{pmatrix} \frac{d}{d\xi}\frac{(1-\xi)}{2} & \frac{d}{d\xi}\frac{(1+\xi)}{2} \end{pmatrix}$$
$$= \frac{2}{L}\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$[K] = A\int_{-1}^{+1} \begin{matrix} -1/L \\ 1/L \end{matrix}\ E\,[-1/L\ \ 1/L]\ J\,d\xi$$

$$= EA\int_{-1}^{+1} \begin{pmatrix} -1/L \\ 1/L \end{pmatrix} < -1/L\ \ 1/L > \ L/2\,d\xi$$

$$= \frac{EAL}{2}\int \begin{pmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{pmatrix} d\xi$$

$$= \frac{EA}{2L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\int d\xi$$

7. **Explain numerical integration and its application to plane stress problems.**

NUMERICAL INTEGRATION AND APPLICATION TO PLANE STRESS PROBLEMS

In the isoparametric formulation of higher order elements we see that the strain-displacement matrix [B] is given by

$$[B] = \frac{du}{dx} = \frac{dN}{dx}[\xi] = \frac{1}{J}\frac{d[N]}{d\xi}$$

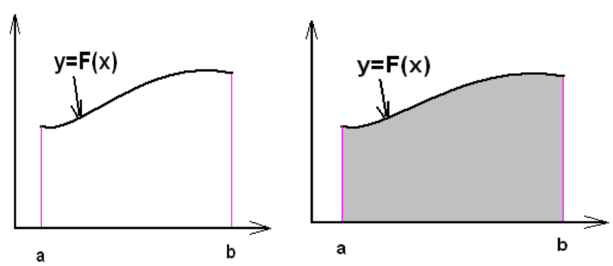
$$= \frac{1}{J}\left(\frac{d}{d\xi}\left(\frac{-\xi+\xi^2}{2} \quad 1-\xi^2 \quad \frac{\xi+\xi^2}{2}\right)\right)$$

Here J = ((-1 + 2ξ)/2 -2ξ (1 + 2ξ)/2)

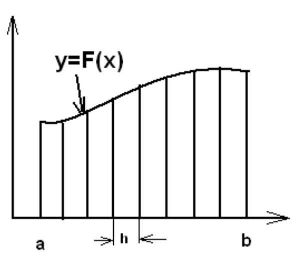
Therefore Matrix [B] is a function of ξ, with polynomials in ξ in its denominator because of the 1/J factor. Hence the equation (A) cannot be integrated to give on the solution. Hence we resort to numerical integration.

So evaluation of integrals of the formb ∫ F(x) dx becomes difficult or impossible in cases where the integrand F has functions of x in both numerator denominator. The basic idea behind whatevernumerical integration technique we may employ is that of obtaining a function P(x) which is both a suitable approximation of F(x) and simple enough to integrate.

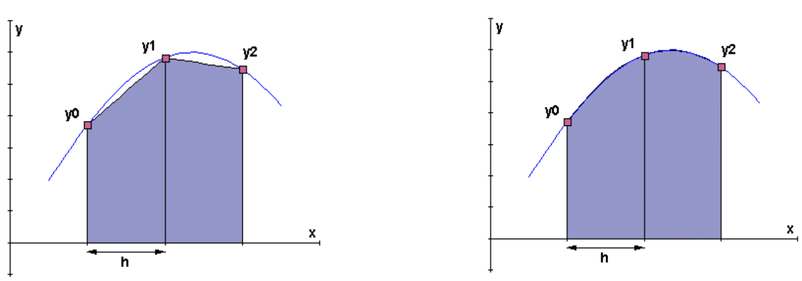
Referring to Fig the variation of F(x) is shown. Evaluation of the Integral $\int_a^b F(x)dx$ will yield the area under the F(x) curve between points x_1 (= a) & x_2 (= b).



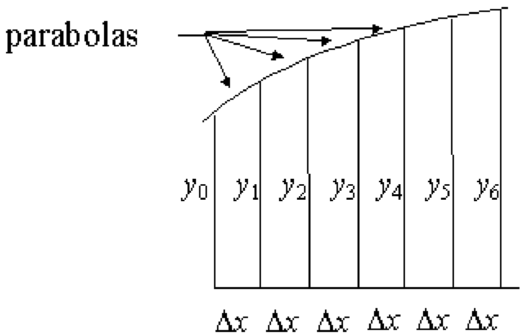
“Trapezoidal rule”,



$$\int_a^b F(x)dx = \frac{h}{2}(y_0 + y_8 + (y_1 + y_2 + \dots\dots y_7))$$



Trapezoidal Rule **Simpsons Rule**



$$F(x)dx = \frac{h}{3}(y_0 + y_8 + 4(y_1 + y_3 + ...y_7) + 2(y_2 + y_4 + ...y_6)$$

8. Explain the Gauss Quadrature in detail.

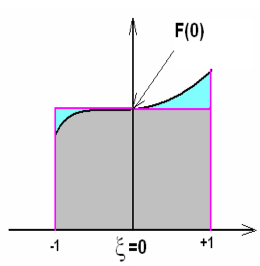
Gauss Quadrature:-Amongst the several schemes available for evaluating the area under the curve F(x) between two points the gauss quadrature method has proved to be most useful for isoparametric elements. As in isoparametric formulation, the limits of the integral are always from –1 to +1, the problem in gauss integration is to evaluate the integral

$$I = \int_{-1}^{+1} F(\xi)\, d\xi.$$

The simplest and probably the crudest way to evaluate the integral is to sample or evaluate F(ξ) at the midpoint of the interval and to multiply this by the length of the element which is ‘2’ [because $\xi_1 = -1$ & $\xi_2 = 1$ & $(\xi_2 - \xi_1) = 2$]

$\therefore \int F(x)\, dx = I = 2 f_i$

This result will be exact only if the actual function happens to be a straight line.



One point formula

We can extend the same to take two sampling points or three etc.Generalization of this relation gives

$$I = \int_{-1}^{+1} F(\xi)\, d\xi = w_1 f_1 + w_2 f_2 + \ldots\ldots w_n f_n = \sum_{i=1}^{n} w_i f(\xi_i)$$

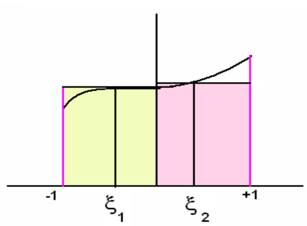
Here w_i is called the 'weight' associated with the i^{th} point and n is the number of sampling points. The Table (1) gives the sampling points and the associated weights (w_i) for Gauss quadrature.

No.of points	Location	Weight W_i
1	$\xi_1 = 0.00000$	2.00000
2	$\xi_1, \xi_2 = \pm 0.57735$	1.000000
3	$\xi_1, \xi_3 = \pm 0.77459$ $\xi_2 = 0.00000$	0.55555 0.88888
4	$\xi_1, \xi_4 = \pm 0.8611363$ $\xi_2 \xi_3 = \pm 0.3399810$	0.3478548 0.6521451

Thus to approximate the integral *I*, the function $f(\xi)$ is evaluated at each of several locations ξ_i, and each $f(\xi_i)$ is multiplied by the approximating weights w. The summation of these products gives the value of the integral. The sampling points are generally located symmetrically with respect to the center of the interval. Symmetrically paired points have the same weight w_i.

9. **Problem: As an example consider the evaluation of the Integral *I* using 2 sampling points i.e. n = 2.**

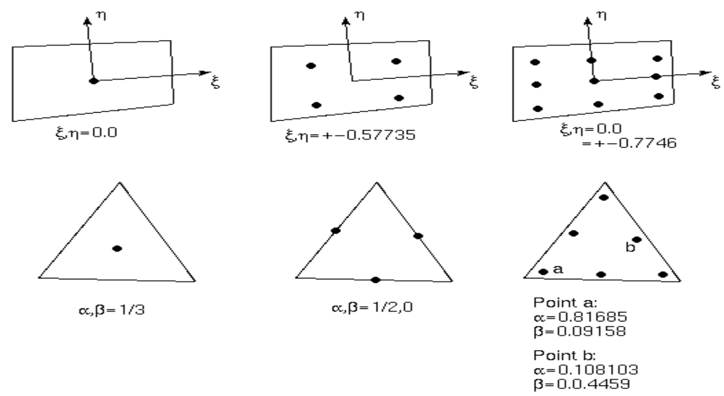
***I* ≈ (1.0) (f at ξ = - 0.577350269189626) + (1.0) (f at ξ = + 0.577350269189626)**



In general if we know that the integral to be evaluated is of order p then the number of sampling points required n is given by the relation

2n-1 = p

The calculated number of sampling points can be rounded off to the nearest integer



10. **Problem: Evaluate the integral** $I=\int_{-1}^{1}(2+x+x^2)\,dx$ **and compare with exact solution.**

Given: Integral,

$$I=\int_{-1}^{1}(2+x+x^2)\,dx \qquad \Rightarrow f(x)=2+x+x^2$$

To Find: The integral I by using Gauss quadrature.

Solution: We know that , the given integrand is a polynomial of order 2.

So, 2n-1 = 2 $\Rightarrow$ 2n =3 $\Rightarrow$ n = 1.5 $\approx$ 2

For two point Gaussian quadrature,

$$x_1=+\sqrt{\frac{1}{3}}=0.577350269 \qquad w_1=1$$

$$x_2=-\sqrt{\frac{1}{3}}=-0.577350269 \qquad w_2=1$$

$$f(x)=2+x+x^2$$

$$f(x_1)=2+x_1+x_1^2$$

$$=2+(0.577350269)+(0.577350269)^2$$

$$f(x_1)=2.9106836$$

$$w_1\ f(x_1)=1\times 2.9106836$$

$$\Rightarrow w_1\ f(x_1)=2.9106836$$

$$f(x_2)=2+x_2+x_2^2$$

$$=2-(0.577350269)+(-0.577350269)^2$$

$$f(x_2)=1.755983$$

$$w_2\,f(x_2)=1\times 1.755983$$

$$w_2\,f(x_2)=1.755983$$

$$w_1 f(x_1) + w_2 f(x_2) = 2.9106836 + 1.755983$$
$$= 4.666666$$

$$\Rightarrow \int_{-1}^{1} (2 + x + x^2)\, dx = 4.666666$$

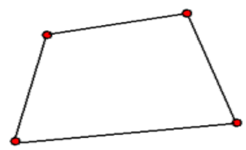
Exact Solution:

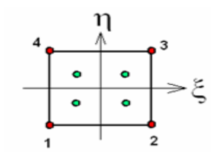
$$\int_{-1}^{1} (2 + x + x^2)\, dx = 2[x]_{-1}^{+1} + \frac{1}{2}[x^2]_{-1}^{+1} + \frac{1}{3}[x^3]_{-1}^{+1}$$

$$= 2[1-(-1)] + \frac{1}{2}[1-(1)] + \frac{1}{3}[1-(-1)]$$
$$= 4.666666$$

11. Problem: Using Gauss Quadrature evaluate the following integral using 1 2 and 3 point Integration.

$$\text{i)} \int_{-1}^{1} \frac{\sin s}{s(1-s^2)}\, ds \quad \text{ii)} \int_{-1}^{1} \frac{\cos^2 s}{s(1-s^2)}\, ds \quad \text{iii)} \int_{-1}^{1} \frac{r^2-1}{(r+3)^2}\, dr$$

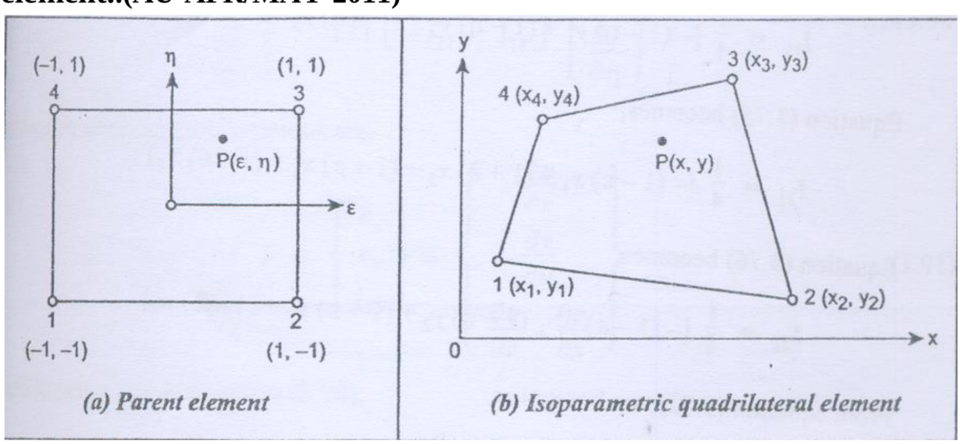




ξ,η=0.57735

$$F(\xi,\eta) = \sum_{i=1}^{n} f(\xi_i, \eta_i)\, w_i w_j$$

12. Derive the element stiffness matrix for four noded isoparametric quadrilateral element..(AU-APR/MAY-2011) (16)



(a) Parent element *(b) Isoparametric quadrilateral element*

The displacement function u for parent rectangular element is given by

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

The displacement function u for isoparametricquadrilateral element is given by

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

Let

$$f = f(x, y)$$

$$f = f[x(\varepsilon,\eta), y(\varepsilon,\eta)]$$

The relationship between natural coordinates and global coordinates can be calculated by using chain rule of partial differentiation.

$$\frac{\partial f}{\partial \varepsilon} = \frac{\partial f}{\partial x} * \frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y} * \frac{\partial y}{\partial \varepsilon}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} * \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} * \frac{\partial y}{\partial \eta}$$

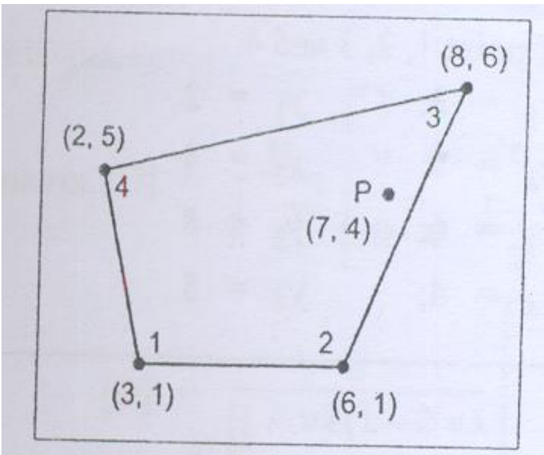
Arranging the above equation in matrix form,

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \varepsilon} & \dfrac{\partial y}{\partial \varepsilon} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

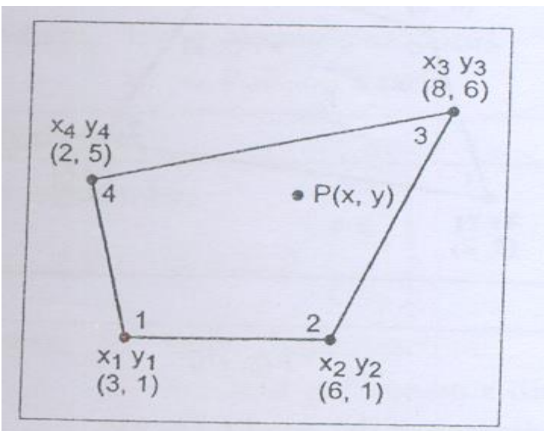
$$\begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

Where J is the Jacobian matrix.

13. For the isoparametric quadrilateral element shown in fig determine the local coordinates of the point P which has Cartesian coordinates (7, 4). (16)



Given:



Cartesian coordinates of point P

X=7; y = 4

Cartesian coordinates of point 1, 2, 3 and 4

X1=3;y1=1

X2=6;y2=1

X3=8;y3=6

X4=2;y4=5

To find:

Local coordinates of point P i.e ε and η

Solution:

We know that
Shape function for quadrilateral element is

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

Cartesian coordinates of point P(x,y)
X= N1x1+N2x2+N3x3+N4x4-------------- 1
Y=N1y1+N2y2+N3y3+N4y4---------------2
Substitute N1, N2, N3 and N4 value in eqn 1

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$7 = \frac{1}{4}[(1-\varepsilon)(1-\eta)*3 + (1+\varepsilon)(1-\eta)*6 + (1+\varepsilon)(1+\eta)*8 + (1-\varepsilon)(1+\eta)*2]$$

$$28 = [(1-\eta-\varepsilon+\eta\varepsilon)3 + (1-\eta+\varepsilon-\eta\varepsilon)6 + (1+\eta+\varepsilon+\eta\varepsilon)8 + (1+\eta-\varepsilon-\eta\varepsilon)2]$$

$$28 = 3-3\eta-3\varepsilon+3\varepsilon\eta+6-6\eta+6\varepsilon-6\varepsilon\eta+8+8\eta+8\varepsilon+8\varepsilon\eta+2+2\eta-2\varepsilon-2\varepsilon\eta$$

$$28 = 19+\eta+9\varepsilon+3\varepsilon\eta$$

$$\eta+9\varepsilon+3\varepsilon\eta = 9 \text{-------}3$$

Substitute N1, N2, N3 and N4 value in eqn 2

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

$$4 = \frac{1}{4}[(1-\varepsilon)(1-\eta)*1 + (1+\varepsilon)(1-\eta)*1 + (1+\varepsilon)(1+\eta)*6 + (1-\varepsilon)(1+\eta)*5]$$

$$16 = [(1-\eta-\varepsilon+\eta\varepsilon)1 + (1-\eta+\varepsilon-\eta\varepsilon)1 + (1+\eta+\varepsilon+\eta\varepsilon)6 + (1+\eta-\varepsilon-\eta\varepsilon)5]$$

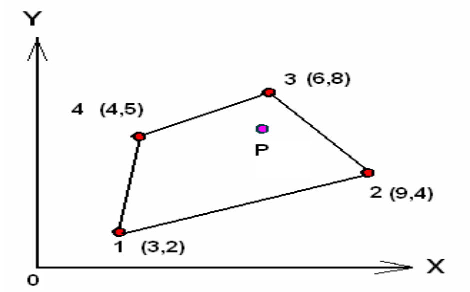
$$16 = 1-\eta-\varepsilon+\varepsilon\eta+1-\eta+\varepsilon-\varepsilon\eta+6+6\eta+6\varepsilon+6\varepsilon\eta+5+5\eta-5\varepsilon-5\varepsilon\eta$$

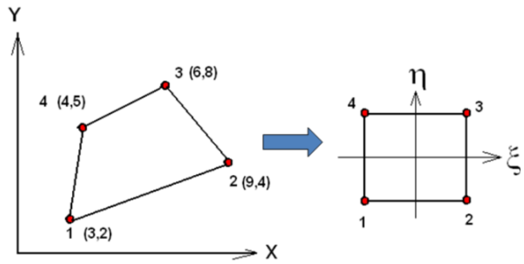
$$16 = 13+9\eta+\varepsilon+\varepsilon\eta$$

$$9\eta+\varepsilon+\varepsilon\eta = 13 \text{--------}4$$

Solving equation 3 and 4
η=0.210587
ε= 0.912545

14. **Problem:Evaluate the Cartesian co-ordinate of the point P which has local co-ordinates ξ = 0.6 and η = 0.8 as shown in the Figure.**





Given: Natural co-ordinates of point P

$\xi = 0.6$ $\eta = 0.8$

Cartesian co-ordinates of point 1,2,3 and 4

$$x_1 = 3; \quad y_1 = 2$$
$$x_2 = 9; \quad y_2 = 4$$
$$x_3 = 6; \quad y_3 = 8$$
$$x_4 = 4; \quad y_4 = 5$$

To Find: The Cartesian co-ordinates of the point P (x,y)

Solution:Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

Substituting the values

$$\Rightarrow N_1(0.6,0.8) = \frac{1}{4}(1-0.6)(1-0.8) = 0.02$$

$$\Rightarrow N_2(0.6,0.8) = \frac{1}{4}(1+0.6)(1-0.8) = 0.08$$

$$\Rightarrow N_3(0.6,0.8) = \frac{1}{4}(1+0.6)(1+0.8) = 0.72$$

$$\Rightarrow N_4(0.6,0.8) = \frac{1}{4}(1-0.6)(1+0.8) = 0.18$$

$$Co-ordinate, x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$
$$= 0.02(3) + 0.08(9) + 0.72(6) + 0.18(4)$$
$$x = 5.82$$
$$Co-ordinate, y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$
$$= 0.02 \times (2) + 0.08(4) + 0.72(8) + 0.18(5)$$
$$y = 7.02$$
$$Co-ordinates\ are\ ((x, y) = (5.82, 7.02))$$

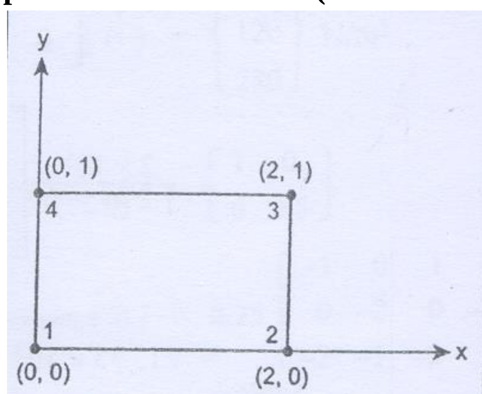
15. **A four noded rectangular element is shown in fig. determine**

1. Jacobian matrix, (4)

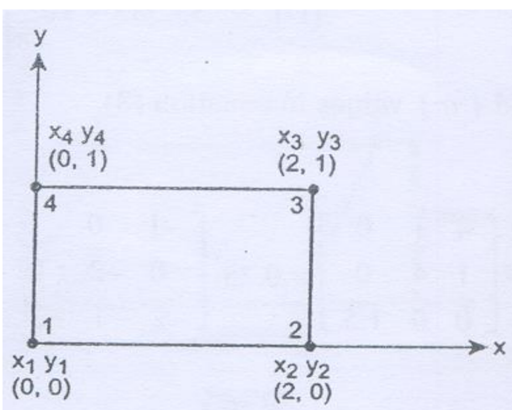
2. Strain – displacement matrix, (8)

3. Element stresses. (4)

Take E = 2*10^{5}N/mm^2, v=0.25, u=[0,0,0.003,0.004,0.006,0.004,0,0]T; ξ=0; η=0. Assume plane stress condition.(AU-APR/MAY-2012)



Given:



Cartesian coordinates of points 1, 2, 3 and 4.

$x_1=0;y_1=0$

$x_2=2;y_2=0$

$x_3=2;y_3=1$

$x_4=0;y_4=1$

Young's modulus, E = $2*10^5$N/mm^2

Poisson's ratio v =0.25

$$displacemrnt, u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

Natural coordinates, ξ=0; η=0.

To find

1. Jacobian matrix
2. Strain-Displacement matrix [B]
3. Element stress, σ.

Solution:

Jacobian matrix for quadrilateral element is given by

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right] \text{----------1}$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right] \text{-----------2}$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 + (1-\varepsilon)x_2 + (1+\varepsilon)x_3 - (1+\varepsilon)x_4\right] \text{------------3}$$

$$J_{22} = \frac{1}{4}\left[-(1-\varepsilon)y_1 + (1-\varepsilon)y_2 + (1+\varepsilon)y_3 - (1+\varepsilon)y_4\right] \text{----------4}$$

Substitute x1, x2, x3, x4, y1, y2, y3, y4, ξ and η value in the above equation 1, 2, 3 and 4

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right] \qquad J_{11} = \frac{1}{4}\left[0+2+2-0\right] \qquad J_{11} = 1$$

$$J_{12}=\frac{1}{4}\left[-(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4\right] \qquad J_{12}=\frac{1}{4}\left[0+0+1-1\right] \qquad J_{12}=0$$

$$J_{21}=\frac{1}{4}\left[-(1-\varepsilon)x_1+(1-\varepsilon)x_2+(1+\varepsilon)x_3-(1+\varepsilon)x_4\right] \qquad J_{21}=\frac{1}{4}\left[0-2+2+0\right] \qquad J_{21}=0$$

$$J_{22}=\frac{1}{4}\left[-(1-\varepsilon)y_1+(1-\varepsilon)y_2+(1+\varepsilon)y_3-(1+\varepsilon)y_4\right] \qquad J_{22}=\frac{1}{4}\left[-0-0+1+1\right] \qquad J_{22}=0.5$$

$$\left[J\right]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \qquad \left[J\right]=\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad \left[J\right]=1*0.5-0 \qquad \left[J\right]=0.5$$

We know that strain displacement matrix for quadrilateral element is,

$$\left[B\right]=\frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}*\frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

Substitute J11, J12, J21, J22, η and ξ value in the above equation.

$$\left[B\right]=\frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix}*\frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\left[B\right]=\frac{1}{0.5*4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$\left[B\right]=\frac{0.5}{0.5*4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$\left[B\right]=0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

We know that

Element stress, σ = [D] [B] {u}

For plane stress condition,

Stress strain relationship matrix, $[D]=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$

$$[D]=\frac{2*10^5}{1-(0.25)^2}\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & \frac{1-0.25}{2}\end{bmatrix}$$

$$[D]=213.3*10^3\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}$$

$$[D]=213*10^3*0.25\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}$$

$$[D]=53.33*10^3\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}$$

Substitute [D], [B] and {u} value in the element stress relation,

$$\{\sigma\}=53.33*10^3\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}*0.25\begin{bmatrix}-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2\\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1\end{bmatrix}\begin{Bmatrix}0\\0\\0.003\\0.004\\0.006\\0.004\\0\\0\end{Bmatrix}$$

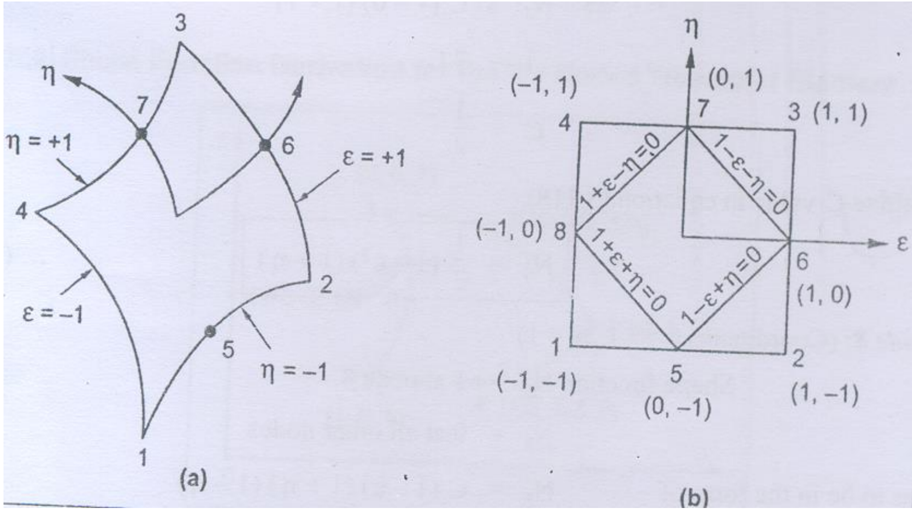
$$\{\sigma\}=13.33*10^3\begin{bmatrix}0+0+(4*0.003)+(-2*0.004)+(4*0.006)+(2*0.004)+0+0\\ 0+0+(1*0.003)+(-8*0.004)+(1*0.006)+(8*0.004)+0+0\\ 0+0+(-3*0.003)+(1.5*0.004)+(3*0.006)+(1.5*0.004)+0+0\end{bmatrix}$$

$$\{\sigma\}=13.33*10^3\begin{Bmatrix}0.036\\0.009\\0.021\end{Bmatrix}$$

$$\{\sigma\}=\begin{Bmatrix}480\\120\\280\end{Bmatrix}N/m^2$$

16. Derive the shape function for eight noded triangular element.(AU-APR/MAY-2014)

Consider a eight noded rectangular element. It belongs to the serendipity family of elements. It consists of eight nodes, which are located on the boundary.



(16)

We know that shape function N_1 = 1 at node 1 and 0 at all other nodes. The natural coordinates of the nodes are indicated in the figure. By following the procedure the shape function can be obtained as,

At node 1:

Coordinates $\xi = -1$, $\eta = -1$

Shape function $N_1 = 1$ at node 1

$N_1 = 0$ at all other nodes

N_1 has to be in the form of $N_1 = C(1-\xi)(1-\eta)(1+\xi+\eta)$

Where C – constant

Substitute $\xi = -1$, $\eta = -1$in equation.

$$N_1 = C(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$1 = -4C$$

$$C = -1/4$$

Substitute C value in the above equation

$$N_1 = C(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

At node 2:

Coordinates $\xi = 1$, $\eta = -1$

Shape function $N_2 = 1$ at node 2

$N_2 = 0$ at all other nodes

N_2 has to be in the form of $N_2 = C(1+\xi)(1-\eta)(1-\xi+\eta)$

Substitute $\xi = 1$, $\eta = -1$in equation.

$$N_2 = C(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$1 = -4C$$

C = -1/4

Substitute C value in the above equation

$$N_2 = C(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

At node 3:

Coordinates $\xi = 1$, $\eta = 1$

Shape function $N_3 = 1$ at node 3

$N_3 = 0$ at all other nodes

N3 has to be in the form of $N_3 = C(1+\xi)(1+\eta)(1-\xi-\eta)$

Substitute $\xi = 1$, $\eta = 1$in equation.

$$N_3 = C(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$1 = -4C$$

C = -1/4

Substitute C value in the above equation

$$N_3 = C(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

At node 4:

Coordinates $\xi = -1$, $\eta = 1$

Shape function $N_4 = 1$ at node 4

$N_4 = 0$ at all other nodes

N_4 has to be in the form of $N_4 = C(1-\xi)(1+\eta)(1+\xi-\eta)$

Substitute $\xi = -1$, $\eta = 1$in equation.

$$N_4 = C(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$1 = -\frac{1}{4}C$$

C = -1/4

Substitute C value in the above equation

$$N_4 = C(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

Now we define N_5, N_6, N_7 and N_8

At node 5:

Coordinates $\xi = 0$, $\eta = -1$

Shape function $N_5 = 1$ at node 5

$N_5 = 0$ at all other nodes

N_5 has to be in the form of $N_5 = C(1-\xi)(1-\eta)(1+\xi)$

Where C – constant

Substitute ξ = -1, η = -1in equation.

$$N_5 = C(1-\xi)(1-\eta)(1+\xi)$$

$$1 = 2C$$

C = 1/2

Substitute C value in the above equation

$$N_5 = C(1-\xi)(1-\eta)(1+\xi)$$

$$N_5 = \frac{1}{2}(1-\xi)(1-\eta)(1+\xi)$$

At node 6:

Coordinates ξ = 1, η = -1

Shape function N_6 = 1 at node 6

N_6 = 0 at all other nodes

N_6 has to be in the form of $N_6 = C(1+\xi)(1-\eta)(1+\eta)$

Substitute ξ = 1, η = 0 in equation.

$$N_6 = C(1+\xi)(1-\eta)(1+\eta)$$

$$1 = 2C$$

C = 1/2

Substitute C value in the above equation

$$N_6 = C(1+\xi)(1-\eta)(1+\eta)$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta)(1+\eta)$$

At node 7:

Coordinates ξ = 1, η = 1

Shape function N_7 = 1 at node 7

N_7 = 0 at all other nodes

N_7 has to be in the form of $N_7 = C(1+\xi)(1+\eta)(1-\xi)$

Substitute ξ = 0, η = 1in equation.

$$N_7 = C(1+\xi)(1+\eta)(1-\xi)$$

$$1 = 2C$$

C = 1/2

Substitute C value in the above equation

$$N_7 = C(1+\xi)(1+\eta)(1-\xi)$$

$$N_7 = \frac{1}{2}(1+\xi)(1+\eta)(1-\xi)$$

At node 8:

Coordinates ξ = -1, η = 1

Shape function N_8 = 1 at node 8

N_8 = 0 at all other nodes

N_8 has to be in the form of $N_8 = C(1-\xi)(1+\eta)(1-\eta)$

Substitute ξ = -1, η = 0 in equation.

$$N_8 = C(1-\xi)(1+\eta)(1-\eta)$$

$$1 = 2C$$

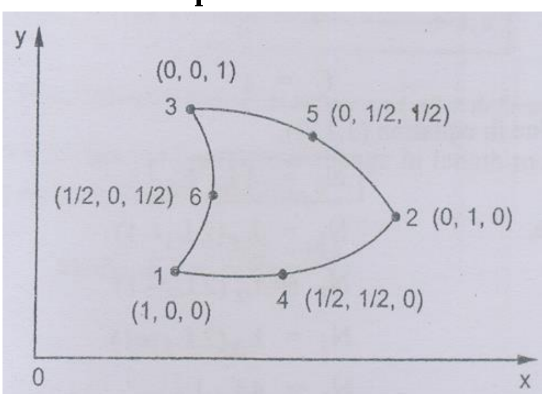
C = 1/2

Substitute C value in the above equation

$$N_8 = C(1-\xi)(1+\eta)(1-\eta)$$

$$N_8 = \frac{1}{2}(1-\xi)(1+\eta)(1-\eta)$$

17. Derive the shape function for six noded triangular element. (16)



Consider a six noded triangular element. It belongs to the serendipity family of elements. It consists of six nodes, which are located on the boundary.

We know that shape function N_1 = 1 at node 1 and 0 at all other nodes. The natural coordinates of the nodes are indicated in the figure. By following the procedure the shape function can be obtained as,

At node 1: Coordinates $L_1 = 1$, $L_2 = 0$, $L_3 = 0$

Shape function N_1 = 1 at node 1

N_1 = 0 at all other nodes

N_1 has to be in the form of $N_1 = CL_1\left(L_1 - \frac{1}{2}\right)$

Where C – constant

Substitute L_1 =1 in equation.

$$N_1 = CL_1\left(L_1 - \frac{1}{2}\right)$$

$$1 = \frac{1}{2}C$$

C = 2

Substitute C value in the above equation

$$N_1 = CL_1\left(L_1 - \frac{1}{2}\right)$$

$$N_1 = 2L_1\left(L_1 - \frac{1}{2}\right)$$

$$N_1 = L_1\left(2L_1 - 1\right)$$

At node 2:

Coordinates $L_1 = 0$, $L_2 = 1$, $L_3 = 0$

Shape function $N_2 = 1$ at node 2

$N_2 = 0$ at all other nodes

N_2 has to be in the form of $N_2 = CL_2\left(L_2 - \frac{1}{2}\right)$

Substitute $L_2 = 1$ in equation.

$$N_2 = CL_2\left(L_2 - \frac{1}{2}\right)$$

`

$$1 = \frac{1}{2}C$$

C = 2

Substitute C value in the above equation

$$N_2 = CL_2\left(L_2 - \frac{1}{2}\right)$$

$$N_2 = 2L_2\left(L_2 - \frac{1}{2}\right)$$

$$N_2 = L_2\left(2L_2 - 1\right)$$

At node 3:

Coordinates $L_1 = 0$, $L_2 = 0$, $L_3 = 1$

Shape function $N_3 = 1$ at node 3

$N_3 = 0$ at all other nodes

N3 has to be in the form of $N_3 = CL_3\left(L_3 - \frac{1}{2}\right)$

Substitute $L_2 = 1$ in equation.

$$N_3 = CL_3\left(L_3 - \frac{1}{2}\right)$$

$$1 = \frac{1}{2}C$$

C = 2

Substitute C value in the above equation

$$N_3 = CL_3\left(L_3 - \frac{1}{2}\right)$$

$$N_3 = 2L_3\left(L_3 - \frac{1}{2}\right)$$

$$N_3 = L_3\left(2L_3 - 1\right)$$

Now we define N_4, N_5 and N_6 at the mid - points

At node 4:

Coordinates $L_1 = 1/2$, $L_2 = 1/2$, $L_3 = 0$

Shape function $N_4 = 1$ at node 4

$N_4 = 0$ at all other nodes

N_4 has to be in the form of $N_4 = CL_1L_2$

Substitute $L_1 = 1/2$, $L_2 = 1/2$ in equation.

$$N_4 = CL_1L_2$$

$$N_4 = C * \frac{1}{2} * \frac{1}{2}$$

$$1 = \frac{1}{4}C$$

C = 4

Substitute C value in the above equation

$$N_4 = CL_1L_2$$

$$N_4 = 4L_1L_2$$

At node 5:

Coordinates $L_1 = 0$, $L_2 = 1/2$, $L_3 = 1/2$

Shape function $N_5 = 1$ at node 5

$N_5 = 0$ at all other nodes

N_5 has to be in the form of $N_5 = CL_2L_3$

Substitute $L_2 = 1/2$, $L_3 = 1/2$ in equation.

$$N_5 = CL_2L_3$$

$$N_5 = C * \frac{1}{2} * \frac{1}{2}$$

$$1 = \frac{1}{4}C$$

C = 4

Substitute C value in the above equation

$N_5 = CL_2L_3$

$N_5 = 4L_2L_3$

At node 6:

Coordinates $L_1 = 1/2$, $L_2 = 0$, $L_3 = 1/2$

Shape function $N_6 = 1$ at node 6

$N_6 = 0$ at all other nodes

N_6 has to be in the form of $N_6 = CL_1L_3$

Substitute $L_2 = 1/2$, $L_3 = 1/2$ in equation.

$N_6 = CL_1L_3$

$$N_6 = C * \frac{1}{2} * \frac{1}{2}$$

$$1 = \frac{1}{4}C$$

C = 4

Substitute C value in the above equation

$N_6 = CL_1L_3$

$N_6 = 4L_1L_3$

18. Evaluate $\int_{-1}^{1}(x^4 + x^2)dx$ by applying 3 point Gaussian quadrature. (16)

Given;

Integral, I= $\int_{-1}^{1}(x^4 + x^2)dx$

$f(x)=x^4+x^2$

To find: evaluate the integral by using Gaussian quadrature with three gauss points.

Solution: we know that, for three point Gaussian quadrature.

$X_1 = \sqrt{\frac{3}{5}} = 0.774596669$

$X_2 = 0$

$X_3 = -\sqrt{\frac{3}{5}} = -0.774596669$

$W_1 = \frac{5}{9} = 0.5555$

$W_2 = \frac{8}{9} = 0.888888$

$W_3 = \frac{5}{9} = 0.5555$

We know that, $f(x) = x^4+x^2$

$f(x)=(x_1)^4+(x_1)^2$

$\boxed{f(x_1)=0.96}$

$W_1\ f(x_1)=0.5555x0.96$

$\boxed{W_1\ f(x_1)=0.5333\ \ldots\ldots..(1)}$

$f(x_2)=x^4+x^2=(0)^4+(0)^2$

$\boxed{f(x_2)=0}$

$W_2\ f(x_2)=0.8888x0$

$\boxed{W_2\ f(x_2)=0\ \ldots\ldots\ldots..(2)}$

$f(x_3)=x_3^4+x_3^2$

$=(-0.774596669)^4+(-0.7774596669)^2$

$\boxed{f(x_3)=0.96}$

$W_3\ f(x_3)=0.55555x0.96$

$\boxed{W_3\ f(x_3)=0.533\ldots\ldots(3)}$

Adding equation (1), (2) and (3),

$W_1f(x_1)+w_2f(x_2)+w_3f(x_3)=0.5333+0+0.5333=1.0666$

Result: $\int_{-1}^{1}\left(x^4+x^2\right)dx=1.06666$

Verification $\int_{-1}^{1}\left(x^4+x^2\right)dx=\left[\frac{x^5}{5}\right]_{-1}^{+1}+\left[\frac{x^3}{3}\right]_{-1}^{+1}$

$$=\frac{1}{5}\left[(1)^5-(-1)^5\right]+\frac{1}{3}\left[(1)^3-(-1)^3\right]$$

$$=\frac{1}{5}[1+1]+\frac{1}{3}[1+1]=1.0666$$

19. Integrate the function $f(r)=1+r+r^2+r^3$ between the limits -1to+1 using,

(i) Exact method. (8)

(ii) Gauss integration method and compare the two results.(8)

(AU-APR/MAY-2014)

Given: function, $f(r)=1+r+r^2+r^3$

To find: evaluate the integral using gauss integration method and compare with method.

Solution: we know that, the given integral is a polynomial of order 3.

So, 2n-1=3

2n=4

N=2

we should use two sampling points,

for to point Gaussian quadrature,

$r_1=+\sqrt{\frac{1}{3}}=0.577350269$

$r_2=-\sqrt{\frac{1}{3}}=-0.577350269$

$w_1=1$

$w_2=1$

$f(r)= 1+r+r^2+r^3$

$f(r_1)= 1+r_1+r_1^2+r_1^3$

$1+0.577350269+(0.5777350269)^2+(0.577350269)^3$

$f(r_1)=2.1031336$

$w_1\ f(r_1)=1x2.1031336$

$\boxed{w_1\ f(r_1)=2.1031336\quad (1)}$

$f(r_2)= 1+r_2+r_2^2+r_2^3$

$1+0.577350269+(-0.5777350269)^2+(-0.577350269)^3$

$f(r_2)=0.5635329$

$w_2\ f(r_2)=1x0.5635329$

$\boxed{W_2\ f(r_2)=0.5635329\quad (2)}$

Applying (1) and (2),

$w_1\ f(r_1)+ w_2\ f(r_2)=2.1031336+0.5635326=2.66666$

$$\int_{-1}^{1}\left(1+r+r^2+r^3\right)dr=\left[r+\frac{r^2}{2}+\frac{r^3}{3}\frac{r^4}{4}\right]_{-1}^{1}$$

$$=\left[r\right]_{-1}^{1}+\frac{1}{2}+\left[r^2\right]_{-1}^{1}+\frac{1}{3}\left[r^3\right]_{-1}^{1}+\frac{1}{4}\left[r^4\right]_{-1}^{1}$$

$$=\left[1-(-1)+\right]\frac{1}{2}\left[(1)^2-(-1)^2\right]+\frac{1}{3}\left[(1)^3-(-1)^3\right]+\frac{1}{4}\left[(1)^4-(-1)^4\right]$$

$$=2+\frac{1}{2}(0)+\frac{1}{3}(1+1)+\frac{1}{4}(0)$$

$$\int_{-1}^{1}\left(1+r+r^2+r^3\right)dr = 2.6666$$

Result: 1. $\int_{-1}^{1}\left(1+r+r^2+r^3\right)dr = 2.6666$ (by gauss integration)

2. $\int_{-1}^{1}\left(1+r+r^2+r^3\right)dr = 2.6666$ (by exact method)

20. Explain some matrix solution technique in detail.

MATRIX SOLUTION TECHNIQUES

<u>Element matrices and vectors for a mapped 2D element</u>

<u>Recall: For each element</u>

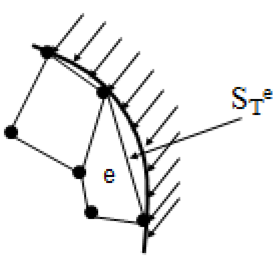
Displacement approximation $\boxed{\underline{u} = \underline{N}\ \underline{d}}$

Strain approximation $\boxed{\underline{\varepsilon} = \underline{B}\ \underline{d}}$

Stress approximation $\boxed{\underline{\sigma} = \underline{D}\underline{B}\ \underline{d}}$

Element stiffness matrix $\boxed{\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\ \underline{B}\, dV}$

Element nodal load vector $\boxed{\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_s}}$



SOLUTIONS TECHNIQUES TO DYNAMIC PROBLEMS

DYNAMIC EQUATIONS OF MOTION

In dynamic problems the displacements, velocities, strains, stresses, and loads are all timedependent. The procedure involved in deriving the finite element equations of a dynamic problem can be stated by the following steps:

Step 1: Idealize the body into E finite elements.

Step 2: Assume the displacement model of element e

Step 3: Derive the element characteristic (stiffness and mass) matrices and characteristic (load) vector.

Step 4:Assemble the element matrices and vectors and derive the overall system equations of motion.

Steps 5 and 6: Solve the equations of motion by applying the boundary and initial conditions.

Once the time history of nodal displacements, Q(t), is known, the time histories of stresses and strains in the elements can be found as in the case of static problems. Special space-time finite elements have also been developed for the solution of dynamic solid and structural mechanics problems.

CONSISTENT AND LUMPED MASS MATRICES

The mass matrix is called "**consistent**" mass matrix of the element. It is called consistent because the same displacement model that is used for deriving the element stiffness matrix is used for the derivation of mass matrix. It is of interest to note that several dynamic problems have been and are being solved with simpler forms of mass matrices. The simplest form of mass matrix that can be used is that obtained by placing point (concentrated) masses mat node points i in the directions of the assumed displacement degrees of freedom.

The consistent mass matrix of the element is given by

$$[m^{(e)}] = \iiint_{V^{(e)}} \rho [N]^T [N] \, dV = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The concentrated masses refer to translational and rotational inertia of the element and are calculated by assuming that the material within the mean locations on either side of the particular displacement behaves like a rigid body while the remainder of the element does not participate in the motion. Thus, this assumption excludes the dynamic coupling that exists between the element displacements, and hence the resulting element mass matrix is purely diagonal and is called the "'**lumped**" mass matrix.

The lumped mass matrix of the element can be obtained

$$[m^{(e)}] = \frac{\rho A l}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consistent Mass Matrix of a Beam Element

For a beam bending element, tile axial displacement degrees of freedom need not be considered and the consistent mass matrix becomes

$$[m^{(e)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

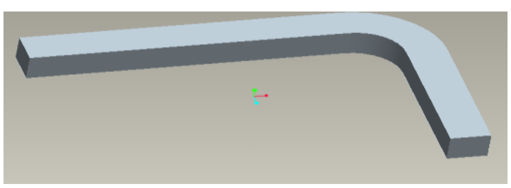
INTRODUCTION TO ANALYSIS SOFTWARE:

21. Explain in detail about the steps in FEA
    Steps in FEA using Pro-Mechanica

- Step 1: Draw part in Pro-Engineer
- Step 2: Start Pro-Mechanica
- Step 3: Choose the Model Type
- Step 4: Apply the constraints
- Step 5: Apply the loads
- Step 6: Assign the material
- Step 7: Run the Analysis
- Step 8: View the results by post-processing

Step 1: Creation of the part

Use Protrusion by Sweep to create this part (bar.prt)



Step 2: Starting Pro-Mechanica

- In Pro-Engineer window, go to Applications→Mechanica to start Pro-Mechanica.
- The part (bar.prt) will be loaded in Pro-Mechanica with a new set of icons for Structural, thermal Analysis

Step 3: Choosing the model type

- In Mechanica menu, select

- Structure → Model →Model Type
- Four different models can be created:
    - 3D Model
    - Plane Stress
    - Plane Strain
    - 2D Axisymmetric
- We will select 3D Model

Step 4: Applying the Constraints

- Create a new constraint by
- Model→ Constraints→ New →Surface
- Give a name for the constraints (fixed_face) and select the surface to be constrained
- Specify the constraints (in our case will be fixed for all degrees of freedom)
- Preview and press Ok

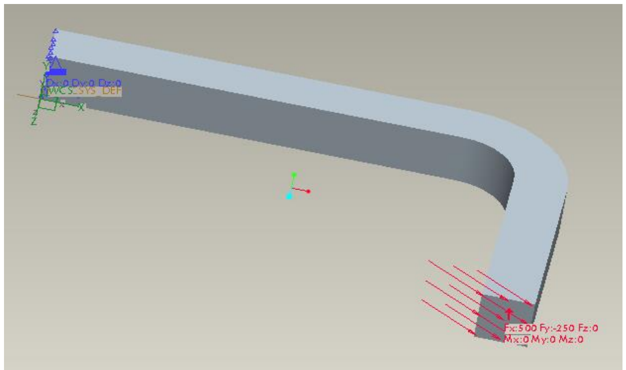
Step 5: Applying the loads

- Similar to Constraints, create a new load by Model → Load→ New → Surface
- Give a name for the applied load (endload)
- Select the surface where the load will be applied
- Specify the loads (Fx:500, Fy:-250, Fz:0)
- Preview and press Ok

Step 6: Assigning the material

- Model → Materials
- A window will pop up with the list of Pro-Mechanica materials. Add the required material and then assign the material to the part.
- Click on Edit if any change in material properties are to be made.
- Press Ok

Modeled part with constraints and loads



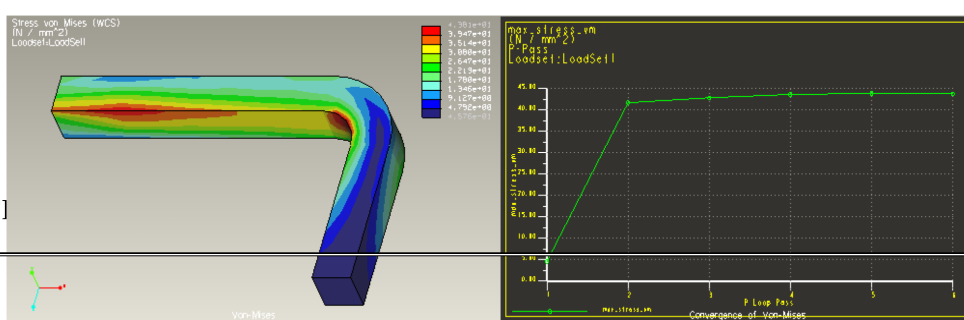
Step 7: Running the Analysis

- In Mechanica menu, select Analysis
- Select File → New Static in “Analysis and Design Studies” dialog box and give a name for the analysis (bar).
- The constraints and loads are automatically loaded.
- In Convergence tab, select Quick Check to check for errors and then select Multi-pass adaptive for the reliable and accurate results. Change the order of the polynomial and percentage of convergence as required.
- Finally, click on Run icon to start the analysis (click on Display Study Status to view the current status and completion of the analysis)

Step 8: Viewing the results

- For post-processing, select Results from Pro-Mechanica window
- A new window will open, and click on “Insert a New Definition” icon. In the dialog box, select the folder where the analysis is saved.
- Select Fringe as Display type, Stress as Quantity and von-mises as the stress component to display
- Similarly, other quantities can be displayed in one window.

Post-processing Results



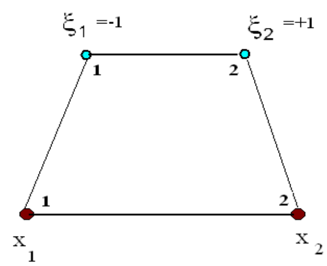
Two Marks Question and Answers.

UNIT-V

1. Define natural co-ordinate systems

A Natural Co-ordinate system is a local co-ordinate system that permits the specification of a point within an element by a set of dimensionless numbers whose absolute magnitude never exceeds unity

i.e. A I Dimensional element described by means of its two end vertices (x_1& x_2) in Cartesian space is represented or mapped on to Natural co-ordinate space by the line whose end vertices ξ_1&ξ_2 are given by –1 & +1 respectively.



2. List some of the advantages of natural co-ordinate systems

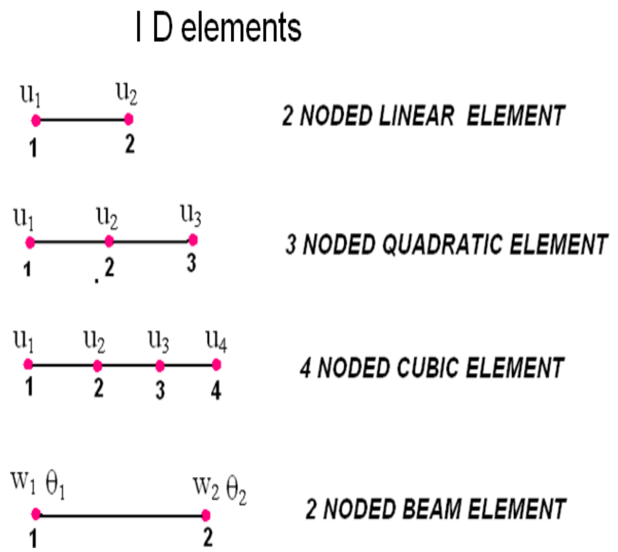
v. It is very convenient in constructing interpolation functions.

vi. Integration involving Natural co-ordinate can be easily performed as the limits of the Integration is always from –1 to +1. This is in contrast to global co-ordinates where the limits of Integration may vary with the length of the element.

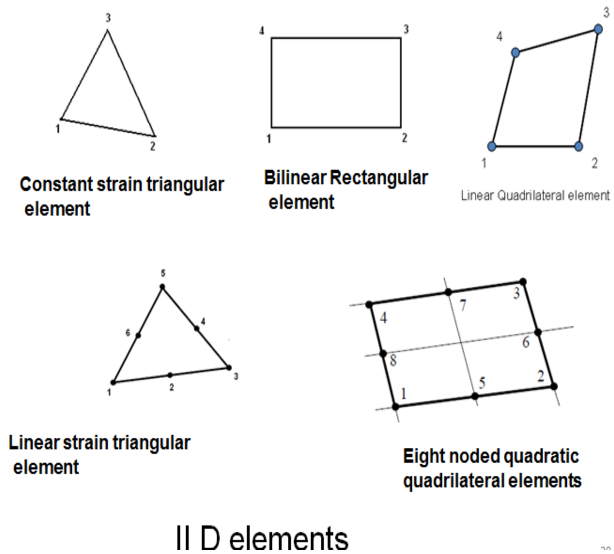
vii. The nodal values of the co-ordinates are convenient number or fractions.

viii. It is possible to have elements with curved sides.

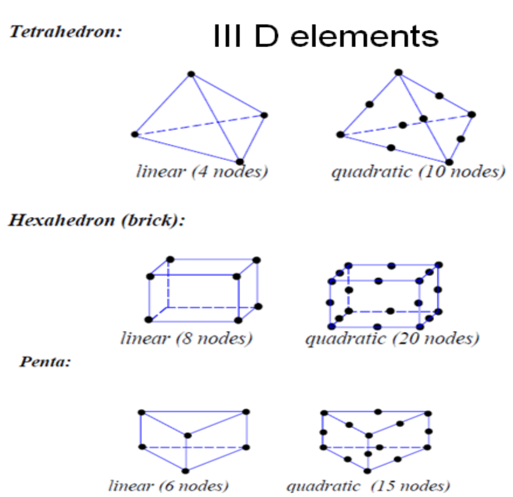
3. List some 1D element.



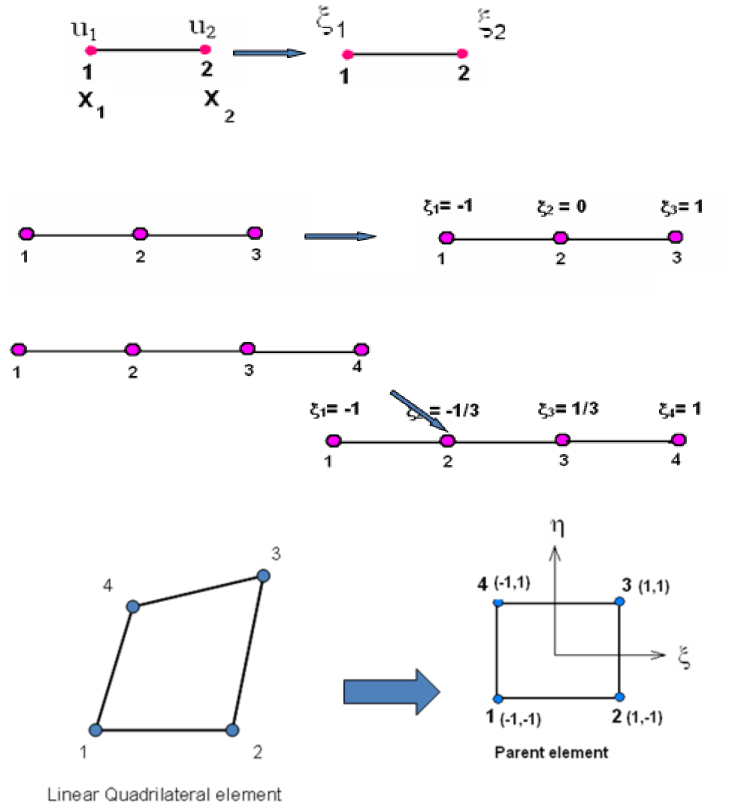
4. List some 2 D element.



5. List some 3D element.



6. How the local coordinates are Conved into natural coordinates



7. Express Lagrangian Interpolation polynomials for rectangular Element: (Natural Co-ordinates)

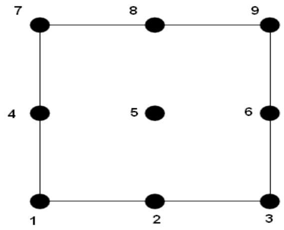
$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$

$$N_2(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$

$$N_4(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$

8. Write the shape function for Nine Noded Quadratic Quadrilateral element



$N_1(\xi,\eta) = N_1(\xi)\, N_2(\eta) = \tfrac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$

$N_2(\xi,\eta) = ½ (1 - \xi^2) (\eta^2-\eta)$

$N_3(\xi,\eta) = ¼ (\xi^2 + \xi) (\eta^2-\eta)$

$N_4(\xi,\eta) = ½ (\xi^2 - \xi) (1 - \eta^2)$

$N_5(\xi,\eta) = (1 - \xi^2) (1 - \eta^2)$

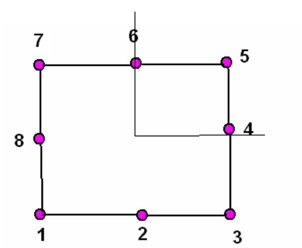
$N_6(\xi,\eta) = ½ (\xi^2 + \xi) (1 - \eta^2)$

$N_7(\xi,\eta) = ¼ (\xi^2 - \xi) (\eta^2 + \eta)$

$N_8(\xi,\eta) = ½ (1 - \xi^2) (\eta^2 + \eta)$

$N_9(\xi,\eta) = ¼ (\xi^2 + \xi) (\eta^2 + \eta)$

9. Write Shape functions for Eight noded quadrilateral element



$N_1(\xi,\eta) = - ¼ (1 - \eta)(1 - \xi) (1 + \xi + \eta)$

$N_2 (\xi,\eta) = ½ (1 – \xi^2)(1 - \eta)$

$N_3 (\xi,\eta) = ¼ (1 + \xi) (1 - \eta) (-1 + \xi -\eta)$

$N_4(\xi,\eta) = ½ (1 - \xi) (1 - \eta^2)$

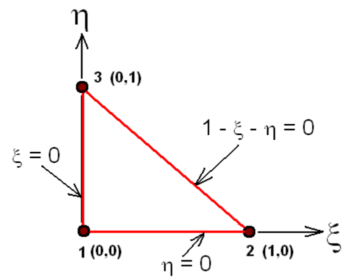
$N_5 (\xi,\eta) = ½ (1 + \xi) (1 - \eta^2)$

$N_6 (\xi,\eta) = ¼ (1 - \xi) (1 + \eta) (-1 - \xi + \eta)$

$N_7 (\xi,\eta) = ½ (1 - \xi^2) (1 + \eta)$

$N_8 (\xi,\eta) = ¼ (1 + \xi) (1 + \eta) (-1 + \xi + \eta)$

10. Express the Shape functions for CST element



$N_1(\xi,\eta) = c\,(1 - \xi - \eta)$ $N_2(\xi,\eta) = c\,(\xi)$

$N_1(0,0) = c\,(1)=1$ $N_2(1,0) = c\,(1) =1$

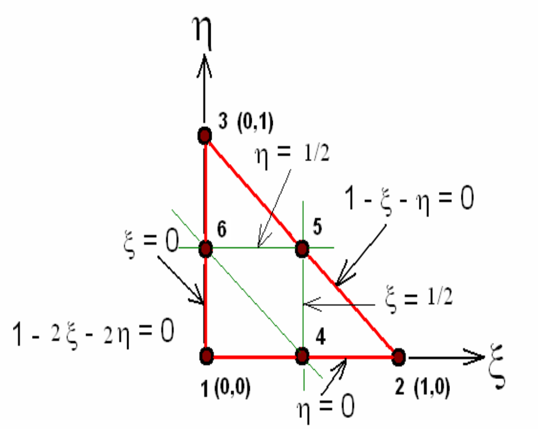
$\therefore c=1$ $\therefore c=1$

$\mathbf{N_1(\xi,\eta) = (1 - \xi - \eta)}$ $\mathbf{N_2(\xi,\eta) = \xi}$

Similarly $N_3= \eta$

11. Express the Shape functions for LST element

$$N_1 = (1-\xi-\eta)(1-2\xi-2\eta)$$
$$N_2 = 2(\xi)(\xi-1/2)$$
$$N_3 = 2(\eta-1/2)(\eta)$$
$$N_4 = 4(\varepsilon)\ (1-\xi-\eta)$$
$$N_5 = 4\xi\eta$$
$$N_6 = 4(1-\xi-\eta)\eta$$



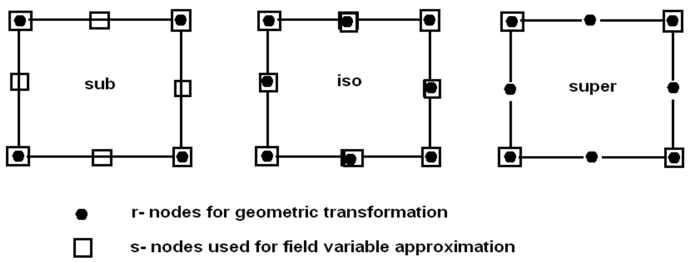
12. What is meant by isoparametric elements?

The element is said to be isoparametric when the shape functions defining the displacement pattern and geometry are the same and are of the same order.

13. What are the three classification of parametric elements?

Depending upon the relationship between these two polynomials elements are classified into three categories as

IV. sub parametric elements $r < s$
V. iso-paramatric elements $r = s$
VI. super-parametric elements $r > s$



14. what is meant by Jacobian of Transformation

Among the 3 cases given above Isoparametric are more commonly used due to their advantages which include the following:

i) Quadrilateral elements in (x,y) coordinates with curved boundaries get transformed to a rectangle of (2 x 2) units in (ξ, η) co-ordinates

ii) Numerical integration is more easily performed as limits of integration vary from –1 to +1 for all elements.

we use the chain rule.

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi}\,\frac{d\xi}{dx} \qquad = \frac{dN_1}{d\xi}\,\frac{1}{dx / d\xi}$$

$$= \frac{dN_1}{d\xi}\,\frac{1}{J} \qquad = J^{-1}\,\frac{dN_1}{d\xi}$$

Here J = dx/ dξ is the 'Jacobian' of transformations from Cartesian space to natural co-ordinate space. It can be considered as the scale factor between the two co-ordinate systems.

Jacobian of transformation for 2 Noded Linear Element

For a 2 Noded element the shape functions are given by

$N_1(\xi) = (1 - \xi)/2$
$N_2(\xi) = (1+\xi)/2$
Now $x = N_1 x_1 + N_2 x_2$
$= (1 - \xi)/2 \; x_1 + (1 + \xi)\; x_2$
$dx/d\xi = J = (-1\, x_1)2 + x_2/2$
$= (x_2 - x_1)/2 = L/2$

Here $(x_2 - x_1)$ represents the length of the element. So the Jacobian of transformation for a 2 noded element is given by L/2

3- Noded Quadratic element:-

$N_1 = -\xi/2\,(1-\xi)$
$N_2 = (1-\xi)\,(1+\xi)$
$N_3 = \xi/2\,(1+\xi)$
$u = N_1 u_1 + N_2 u_2 + N_3 u_3$ &
$x = N_1 x_1\ N_2 x_2 + N_3 x_3$

$$J = \frac{dx}{d\xi} = \begin{pmatrix} \frac{-1+2\xi}{2} & -2\xi & \frac{1+2\xi}{2} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

15. Write the Stiffness Matrix for a 2 Noded Axial Element

$$[K] = \int_0 B^T D\, BAdx$$

$$[B] = \frac{du}{dx} = \frac{dN}{dx} = \frac{1}{J}\frac{dN}{d\xi}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} \end{pmatrix}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{d}{d\xi}\frac{(1-\xi)}{2} & \frac{d}{d\xi}\frac{(1+\xi)}{2} \end{pmatrix}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$[K] = A\int_{-1}^{+1} \begin{matrix} -1/L \\ 1/L \end{matrix}\ E\,[-1/L\ 1/L]\ J\, d\xi$$

$$= EA\int_{-1}^{+1} \begin{pmatrix} -1/L \\ 1/L \end{pmatrix} < -1/L\ \ 1/L > L/2\, d\xi$$

$$= \frac{EAL}{2}\int \begin{pmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{pmatrix} d\xi$$

$$= \frac{EA}{2L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\int d\xi$$

16. What is meant by Gauss Quadrature?

Amongst the several schemes available for evaluating the area under the curve F(x) between two points the gauss quadrature method has proved to be most useful for isoparametric elements. As in isoparametric formulation, the limits of the integral are always from –1 to +1, the problem in gauss integration is to evaluate the integral

$$I = \int_{-1}^{+1} F(\xi)\, d\xi.$$

17. Tabulate the sampling points and the associated weights (w_i) for Gauss quadrature.

No.of points	Location	Weight W_i
1	$\xi_1 = 0.00000$	2.00000
2	$\xi_1, \xi_2 = \pm 0.57735$	1.000000
3	$\xi_1, \xi_3 = \pm 0.77459$ $\xi_2 = 0.00000$	0.55555 0.88888
4	$\xi_1, \xi_4 = \pm 0.8611363$ $\xi_2 \xi_3 = \pm 0.3399810$	0.3478548 0.6521451

18. What is the purpose of isometric elements?

It is difficult to represent the curved boundaries by straight edges finite elements. A large number of finite elements may be used to obtain reasonable resemblance between original body and the assemblage. In order to overcome this drawback, isoparametric elements are used, for problems involving curved boundaries, a family of elements known as "isoparametric elements" are used.

19. Write down the shape functions for 4 noded rectangular elements using natural co-ordinates system. (AU-APR/MAY-2011)

Shape functions: $N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$

$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$

$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$

$N4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$

Where ε and η are natural co-ordinates.

20. Write down the Jacobin matrix for four noded quadrilateral elements. (AU-APR/MAY-2012).

Jacobin matrix, $[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

Where, $J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1-\eta)x_4\right]$

$J_{22} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1-\eta)y_4\right]$

$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 + (1-\varepsilon)x_2 + (1+\varepsilon)x_3 - (1-\varepsilon)x_4\right]$

$J_{22}=\frac{1}{4}\left[-(1-\varepsilon)y_1+(1-\varepsilon)y_2+(1+\varepsilon)y_3-(1-\varepsilon)y_4\right]$

Where ε and η are natural co-ordinates.

$x_1,x_2,x_3,x_4,y1,y_2,y_3,y_4$ are Cartesian co-ordinates.

21. Write down the stiffness equation for four nodedisoparametric quadrilateral element.

Stiffness matrix, $[K]=t\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]X|J|X\partial\varepsilon X\partial\eta$

Where, t= thickness of the element

$|J|$=determinant of the Jacobin

ε,η =natural coordinates.

[B]=strain-displacement matrix.

[D]=stress-strain relationship matrix.

22. Write down the element force vector equation for four noded quadrilateral elements.

Force vector, $\{F\}_e=[N]^T\begin{Bmatrix}F_x\\F_y\end{Bmatrix}$

Where, N-is the shape function

F_x-is a load or force on x-direction

F_y-is a force on y-direction.

23. Write down the Gaussian quadrature expression for numerical integration.

Gaussian quadrature expression:

$$\int_{-1}^{1}f(x)dx=\sum_{i=1}^{n}W_i\,f(x_i)$$

Where, W_i=weight function

$f(x_i)$-values of the function at pre-determined sampling points.

24. Define super parametric element. (AU-APR/MAY-2013)

If the number of nodes used for defining the geometry is more than number of nodes used for defining the displacement, then it is known as super parametric element. The super parametric elements are quadratic element which has the many number of nodes to get the nearest possible solution for the problem. The defined structure or the boundary of the defined problem.

25. What is meant by sub-parametric element?

If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacement, then it is known as sub-parametric element. Sub parametric elements are simple element which are divided when the elements are giving solution possible functions. The functions are the difference between the original value and the find value from the problem.

Generalized coordinates approach to nodal approximations – difficulties

26. What is meant by isoparametric element? (AU-APR/MAY-2014)

If the number of nodes used for defining the geometry is same as number of nodes used for defining the displacement, the n it is known as isoparametric element. Isoparametric elements of simple shapes expressed in natural coordinate system known as master elements are the transformed shapes of some arbitrary curved sided actual elements expressed in Cartesian coordinate system.

27. Is beam element an isoparametric element?

Beam element is not an isoparametric element since the geometry and displacement defining by different order interpolation functions. Isoparametric elements of simple shapes expressed in natural coordinate system known as master elements are the transformed shapes of some arbitrary curved sided actual elements expressed in Cartesian coordinate system.

28. What is the difference between natural co-ordinate and simple natural co-ordinate?

A natural co-ordinate is one whose value lies between zero and one.

Examples L_2=x/l; L_1=(l-x/l)

Area co-ordinates;

$$L_1=A_1/A;\ L_2=A_2/A;\ L_3=A_3/A$$

A simple natural co-ordinate is one whose value lies between -1 and +1.

29. Give examples for essential (forced or geometric) and non-essential (natural) boundaryconditions.

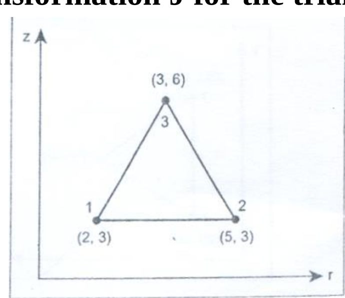
Essential boundary condition:

The geometric boundary conditions are displacement. Slope, etc.

Non-essential boundary condition:

The natural boundary conditions are bending moment, shear force, etc.

30. Calculate the jacobian of the transformation J for the triangular element shown in fig.



Solution:

r_1=2; Z_1=3

r_2=5; Z_2=3

r_3=3; Z_3=6

$$J=\begin{bmatrix} r_1-r_3 & z_1-z_3 \\ r_2-r_3 & z_2-z_3 \end{bmatrix}=\begin{bmatrix} -1 & -3 \\ 2 & -3 \end{bmatrix}$$

$|J|$=3+6=9 units.

31. Write short note on isoparametric element formulation. (AU-APR/MAY-2011)

Isometric formulation:

The principal concept of isoparametric finite element is to express the element coordinates and element displacements in the form of interpolations using the natural coordinate system of the element. These isoparametric elements of simple shapes expressed in natural coordinate system known as master elements are the transformed shapes of some arbitrary curved sided actual elements expressed in Cartesian coordinate system.

iso-parametric elements – structural mechanics applications in 2-dimensions

32. Differentiate isoparametric, super parametric and sub parametric elements.

Isoparametric:

For an element if the geometry and field variables are described by the same shape function of equal order.

Super parametric:

If the order of the shape functions for describing the geometry is more than that of the describing field variable.

Parametric element:

If the order of the shape function describing the geometry is less than that for describing field variable.

33. Define higher order element. (AU-APR/MAY-2010)

Higher order element:

For any element if the interpolation polynomial is of order two or more the element is known as higher order element. In higher order element the field variable variation is non-linear. Also it may be a complex or multiplex element. In higher order element some secondary nodes are produced in addition to the primary nodes in order to match the number of nodal degrees of freedom with the number of polynomial coefficients in the polynomial interpolation.

34. Write in brief about gauss – quadrature method.

Gauss-quadrature method:

Gauss quadrature is thesimple integration method for the definite integrals. It includes some specific functions like weight functions and some sampling points called Gauss-point through which the approximation method has been carried out.

For example

$$I = \int_a^b f(x)dx$$

35. Distinguish between geometric and non – linearity.

Geometry:

Geometry defines the specified structure of any object which gives the boundary of the element in a easy way. Geometry defines structured problems with known quantity and values.

Non – linearity:

The non –structural problems are comes under the non – linearity which gives the geometry by dividing the element in specified structures like triangular, rectangular or quadrilateral elements.

36. What are the differences between implicit and explicit methods of integration?

Implicit methods of integration:

The indirect method of integration is known as implicit method of integration which defines the complicated problem for which we are not able to get the nearest solution through direct method so the implicit method of integrations are used to find the values of such problems.

Explicit methods of integration:

The direct method of integration which gives direct solution and are analysed the direct boundary value problems.

37. Writeshortnoteononepoint andtwopoint Gaussian quadratureapproach.

Gaussian point in Gauss quadrature approach:

In Gauss quadrature method the Gaussian point will be located at equal distances from the origin in the opposite direction. They are symmetrically located in all the element like rectangle or curved elements and are specified by the integral. The distance of the Gauss point from the origin and the values of weight function is located upto five points (n = 1, 2, 3, 4, 5).

Elasticity equations –stress strain relations – plane problems of elasticity – element equations

38. What ismeant bytwo dimensional vectorvariable problem?

Two dimensional vector variable problem:

In vector variable problem the field variable must be described by its magnitude and direction of action in order to get complete information and for further process. In this problem vector quantity is resolved into components parallel to the coordinate axes and these components are treated as the unknown quantities. There will be two unknown parameter at any node in the vector variable problem.

39. What problem aretreated astwo dimensional vector variableproblem?

Problems treated as two dimensional vector variable:

In general the structural problems are treated as the two dimensional vector variable problems for which the nodal displacements due to applied load are resolved into components and processed further. The problems are divided by the system according to the functions as simplex, complex and multiplex elements.

Assembly – need for quadrature formule – transformations to natural coordinates

40. Specify the various elasticity equations.

Elasticity equations:

The equation relating the normal stresses and shear stresses with linear strains and shear strains are known as elasticity for Plane stress condition

$$\sigma_x = \frac{E}{(1-\mu^2)}(e_x + \mu e_y)$$

$$\sigma_y = \frac{E}{(1-\mu^2)}(\mu e_x + e_y)$$

$$\tau_{xy} = \frac{E}{(1-\mu^2)}\left(\frac{1-\mu}{2}\right)\gamma_{xy}$$

for plane strain condition

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)}[(1-\mu)e_x + \mu e_y]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)}(\mu e_x + (1-\mu)e_y)$$

$$\tau_{xy} = \frac{E}{(1+\mu)(1-2\mu)}\left(\frac{1-2\mu}{2}\right)\gamma_{xy}$$

41. Differentiate plane stress with plane strain for two dimensional element.

Plane stress:

A state of plane stress is said to exist when the elastic body is very thin and there is no load applied in the coordinate direction parallel to the thickness.

Example: a ring press – fitted on a shaft is a plane stress problem.

Plane strain:

A state of plane strain occurs in members that are not free to expand in the direction perpendicular to the plane of applied loads.

Example: in a long body of uniform cross-section subjected to transverse loading along its length a small thickness in the loaded area can be treated as plane strain problem.

42. Write short note on principal stresses. (AU-NOV/DEC-2010)

Principal stresses:

The principal stresses are the components of the stress tensor when the basis is changed in such a way that the shear stress components become zero. To find the principal stresses in two dimensions, we have to find the angle θat which $\sigma'_{12} = 0$. This angle is given by

$$\sigma_1 = \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

$$\sigma_2 = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

43. What do you meant by isoparametric representations?

Isoparametric representations:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix},$$

$$u_x = u_{x1} N_1^e + u_{x2} N_2^e + u_{x3} N_3^e = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3,$$

$$u_y = u_{y1} N_1^e + u_{y2} N_2^e + u_{y3} N_3^e = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3.$$

Gaussian quadrature –plane strain and axisymmetric applications

44. What is constitutive relationship?

Constitutive relationship:

Constitutive relationship is the stress strain relationship in the element.

$$[D]=\frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix}(1-\mu) & \mu & \mu & 0\\ \mu & (1-\mu) & \mu & 0\\ \mu & \mu & (1-\mu) & 0\\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right)\end{bmatrix}$$

E – Young's modulus

45. Define frequency of vibration.

It is the number of cycles described in one second. Unit is HZ

46. Define damping ratio.

It is define as the ratio of actual damping coefficient (c) to the critical damping coefficient (c_c)

Damping ratio $\varepsilon = \frac{c}{c_c} \quad = \quad \frac{c}{2m\omega_n}0$

47. What is meant by longitudinal vibration?

When the particles of the shaft or disc moves parallel to the axis of the shaft, then the vibrations are known as longitudinal vibration.

48. What is meant by transverse vibration?

When the particles of the shaft or disc moves approximately perpendicular to the axis of the shaft, then the vibrations are known as transverse vibration.

49. Define magnification factor.

The ratio of the maximum displacement of the forced vibration (x_{max}) to the static deflecyion under the static force (x_0) is known as magnification factor.

50. Write down the expression of longitudinal vibration of bar element.

Free vibration equation for axial vibration of bar element is

$$[K]\{u\} \ = \ \omega^2[m]\{u\}$$

Where, u - displacement

$[K]$– stiffness matrix

$$[K] - \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

ω – natural frequency

$[m]$ – mass matrix

Lamped $[m] = \frac{\rho Al}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Consistent $[m] = \frac{\rho AL}{2}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

51. Write down the expression of governing equation for free axial vibration of rod.

The governing equation for free axial vibration of rod is given by,

$$AE\ \frac{\partial^2 u}{\partial x^2} = \rho A\ \frac{\partial^2 u}{\partial t^2}$$

Where, E – Young's modulus,

A – cross section area

ρ – density

52. Write down the expression of governing equation for transverse vibration of beam

The governing equation for free transverse vibration of a beam is

$$EI\ \frac{\partial^4 v}{\partial x^4} + \rho A \frac{\partial^2 v}{\partial t^2} = 0$$

Where, E –Young's modulus

I –moment of inertia

ρ –density

A –cross sectional area

53. Write down the expression of transverse vibration of beam element.

Free vibration equation for transverse vibration of beam element is,

$$[K]\{u\} = \omega^2 [m]\{u\}$$

Where, $[K]$ = stiffness matrix for beam element

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$[m]$ = mass matrix

$$[m] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \textit{for consistent mass matrix}$$

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \textit{for lumped mass matrix}$$

54. What are the types of Eigen value problem?

There are essentially three groups of method of solution,

1. Determinant based methods
2. Transformation based methods
3. Vector iteration methods

55. State the principle of superposition.

It states that for linear system, the individual responses to several disturbances or driving function can be superposed on each other to obtain the total response of the system.

56. Define resonance.

When the frequency of external force is equal to the natural frequency of a vibration body, the amplitude of vibration becomes excessively large. This phenomenon is known as resonance.

57. Define Dynamic Analysis.

When the inertia effect due to the mass of the component is also considered in addition to the externally applied load, then the analysis is called dynamic analysis.

58. What are methods used for solving transient vibration problem?

There are two methods for solving transient vibration problem. They are:

- ✓ Mode superposition method
- ✓ Direct integration method.

59. Write down the expression for undamped system of Direct Integration Method in Central Difference Method.

For an undamped system,

$$\frac{1}{(\Delta t)^2}[m]\{X_{n+1}\} = F(t_n) - \left[[K] - \frac{2}{(\Delta t)^2}[m]\right]\{X_n\} - \frac{1}{(\Delta t)^2}[m]\{X_{n-1}\}$$

60. State the two difference between direct and iterative methods for solving system of equation.

	Direct Method	Iterative Method
(i)	It given exact value	It gives only approximate solution
(ii)	Simple, take less time	Time consuming and labourious
(iii)	Determine all the roots at the same time	Determine only one root at the time

61. Define linear dependence and independence of vectors.

Linear dependence: The vectors $X_1, X_2, \ldots\ldots . X_n$ are said to be linearly dependent if there exist scalars $\lambda_1, \lambda_2, \ldots . . \lambda_n$ (not all zero) such that,

$$X_1 \lambda_1 + X_2 \lambda_2 + \ldots\ldots\ldots + X_n \lambda_n = 0$$

Independence: The vector X_1, $X_2, \ldots\ldots\ldots . X_n$ are said to be linearly independent if $\lambda_1 X_1 + \lambda_2 X_2 + \ldots\ldots\ldots\ldots\ldots.. + \lambda_n X_n$ is equal to zero such that

$$\lambda_1 = 0, \lambda_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. = \lambda_n$$

62. Show that the matrix, $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Solution: $[A] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\therefore \quad [A]^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\therefore \quad [A]^T[A] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{bmatrix} \cos.^2\theta + \sin.^2\theta & 0 \\ 0 & \cos.^2\theta + \sin.^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

∴ A is orthogonal.

63. Prove that the vector (1,4,-2),(-2,1,3) and (-4,11,5) are linearly dependent.

Solution: $A = \begin{vmatrix} 1 & 4 & -2 \\ -2 & 1 & 3 \\ -4 & 11 & 5 \end{vmatrix}$ $|A| = \begin{vmatrix} 1 & 4 & -2 \\ -2 & 1 & 3 \\ -4 & 11 & 5 \end{vmatrix}$

= 1(5-33)-4(-10+12)-2(-22+4) $\quad |A| = 0$